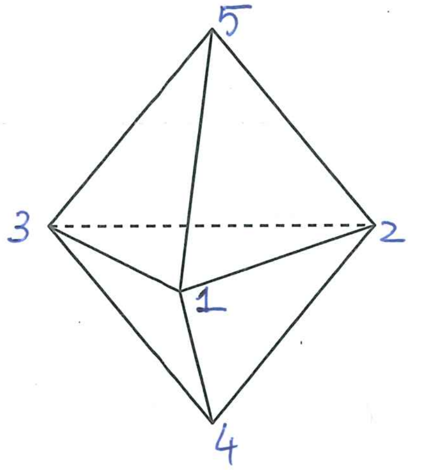
Name and Surname:
Student Number:

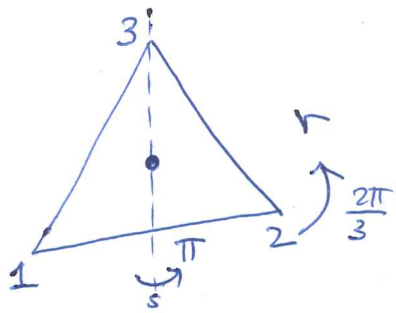
Math 466 - Fall 2019 - METU

Midterm 1 - October 23 - 17:40 - 120 minutes

1. Glue two tetrahedron together so that they have a triangular face in common. Find all the rotational symmetries of this new solid. Label the vertices of this new solid 1 to 5. Using these labels, write a group isomorphism from the rotational symmetry group of this new solid to a subgroup of S_5.



Recall that the rotational symmetries of an equilateral triangle is given by $D_6 = \langle r, s \mid r^3 = e = s^2,\ srs = r^2 \rangle$



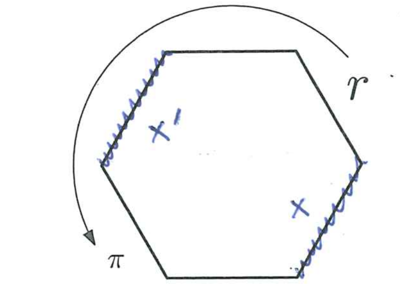
Let G be the group of rotational symmetries of this new solid. Any $g \in G$ must leave the glued face fixed. On the other hand any element of D_6 gives a rotational symmetry of this new solid. More precisely, we have an isomorphism $G \cong D_6$. For each $h \in D_6$, denote the corresponding element of G with $\overline{h}$.

Consider the map
$$\begin{aligned} \varphi : G &\longrightarrow S_5 \\ \overline{e} &\longmapsto (1) \\ \overline{r} &\longmapsto (123) \\ \overline{r^2} &\longmapsto (132) \\ \overline{s} &\longmapsto (12)(45) \\ \overline{rs} &\longmapsto (13)(45) \\ \overline{r^2 s} &\longmapsto (23)(45) \end{aligned}$$

which is induced by the above labeling. The map φ is a homomorphism by construction. Moreover it is injective. Therefore G is isomorphic to a subgroup of S_5.

2. Verify the orbit-stabilizer formula in each example for some x.

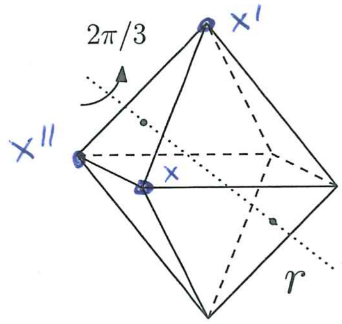
(a) Let X be the set of edges of a regular hexagonal plate and let r be the counterclockwise rotation of the plate by π. Regard r as a permutation of X. Consider the group action $\varphi : \mathbb{Z}_{12} \to \langle r \rangle$ given by the formula $m \mapsto r^m$.



We have $G(x) = \{x, x'\}$ and $G_x = \{0, 2, 4, 6, 8, 10\}$, a subgroup of $\mathbb{Z}_{12}$. Now

$$|\mathbb{Z}_{12}| = 12 = 2 \cdot 6 = |G(x)|\,|G_x|$$

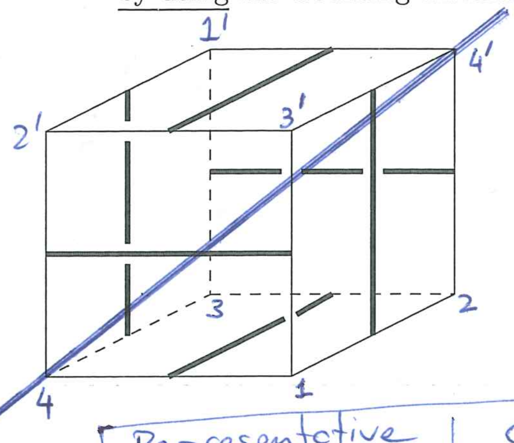
(b) Let X be the set of vertices of an octahedron and let r be a rotation by $2\pi/3$ about an axis passing through the centers of two opposite faces. Regard r as a permutation of X. Consider the group action $\varphi : \mathbb{Z}_{12} \to \langle r \rangle$ given by the formula $m \mapsto r^m$.



We have $G(x) = \{x, x', x''\}$ and $G_x = \{0, 3, 6, 9\}$, a subgroup of $\mathbb{Z}_{12}$. Now

$$|\mathbb{Z}_{12}| = 12 = 3 \cdot 4 = |G(x)|\,|G_x|$$

3. Consider a cube whose each face is bisected with a stripe as shown below so that no two of stripes meet. The bisected cube is decorated by painting each half of the subdivided faces blue, red or green. Determine the number of decorated cubes by using the Counting Theorem.



Even though the rotational symmetry group of the cube is isomorphic to S_4, this solid has A_4 as the symmetry group.

s and s^2

$\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively

Representative of the conjugacy class	Size of the conjugacy class	$\lvert X^g \rvert$
(1)	1	3^{12}
$(123) \rightsquigarrow s$	4	3^4
$(132) \rightsquigarrow s^2$	4	3^4
(12) (34)	3	3^6

all possible colourings

the front and the right face can be painted with 4 options

rotation about the axis through the centers of bottom and top face by π

top and bottom face must be painted with a single colour the front and the right face can be painted with 4 options.

Counting Theorem $\Longrightarrow$ # of decorated cubes $= \frac{1}{12}\left(3^{12}+4\cdot 3^4+4\cdot 3^4+3\cdot 3^6\right) = 44523$

Hint: The conjugacy classes of A_4 are represented by $\{(1)\}, \{(123)\}, \{(132)\}$, and $\{(12)(34)\}$.

4. Let G act on X. State the definitions of $G(x)$ and G_x. Show that the points in the same orbit have conjugate stabilizer groups.

$G(x) = \{gx \mid g \in G\}$ and $G_x = \{g \in G \mid gx = x\}$

Suppose x and y belong to the same orbit, say $G(z)$. Then $g_1 x = z = g_2 y$ for some $g_i \in G$. It follows that $gx = y$ for $g = g_2^{-1} g_1 \in G$.

Claim: $g G_x g^{-1} = G_y$

Pick $h \in G_x$ (i.e $hx = x$). Then

$$\begin{aligned} ghg^{-1}y &= ghg^{-1}gx \\ &= ghx \\ &= gx \\ &= y \end{aligned}$$

Thus $ghg^{-1} \in G_y$, or $g G_x g^{-1} \subseteq G_y$. Switching the roles of x and y, we obtain $g^{-1} G_y g \subseteq G_x$. This proves the claim.

As an example, consider the natural group action of S_4 on the set $X = \{1, 2, 3, 4\}$ given by $\sigma \cdot x = \sigma(x)$. Show that 1 and 2 are in the same orbit. Compute G_1 and G_2 and verify that G_1 and G_2 are conjugate subgroups of S_4.

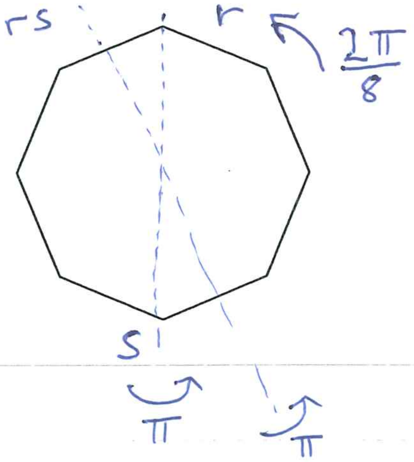
Observe that $(1)(1) = 1$ and $(12)(1) = 2$. Thus 1 and 2 are in the same orbit. Indeed the action is transitive and there is a single orbit $\{1, 2, 3, 4\}$.

We have $G_1 = \{\sigma \in S_4 : \sigma(1) = 1\}$
$= \{(1), (23), (24), (34), (234), (243)\}$

and $G_2 = \{\sigma \in S_4 : \sigma(2) = 2\}$
$= \{(1), (13), (14), (34), (134), (143)\}$

For $\tau = (12)$, we have $\tau G_1 \tau^{-1} = G_2$. Thus G_1 and G_2 are conjugate subgroups of S_4.

5. Decorate a regular plate with 8 edges by colouring each edge blue, red or green. Determine the number of decorated plates by using the Counting Theorem.



$|X| = 3^8$ and $G \cong D_{16}$

Representative of the conjugacy class	Size of the conjugacy class	$\lvert X^g \rvert$
e	1	3^8
r	2	3
r^2	2	3^2
r^3	2	3
r^4	1	3^4
s	4	3^4
rs	4	3^5

Note that $|X^g|$ differs for $g=s$ and $g=rs$. This is because s fixes no edges whereas rs fixes two edges.

By counting theorem, the number of orbits is

$$\frac{1}{16}\left(3^8+2\cdot3+2\cdot3^2+2\cdot3+3^4+4\cdot3^4+4\cdot3^5\right)$$

$$=498$$

Hint: The rotational symmetry group is the dihedral group with 16 elements: $\langle r, s : r^8 = e = s^2, srs^{-1} = r^{-1}\rangle$. The conjugacy classes of this group are $\{e\}, \{r^4\}, \{r, r^7\}, \{r^2, r^6\}, \{r^3, r^5\}, \{s, r^2s, r^4s, r^6s\}$, and $\{rs, r^3s, r^5s, r^7s\}$.