Name and Surname:

Math 466 - Fall 2019 - METU

Final Exam

1. True or False?

(a) If an isometry of $\mathbb{R}^2$ interchanges distinct points P and Q, then it fixes the midpoint of the line segment PQ.

TRUE: Let f be an isometry such that $f(P)=Q$ and $f(Q)=P$. Suppose that M is the midpoint of PQ and set $M'=f(M)$. We have $|PM|=|MQ|$. On the other hand $|QM'|=|M'P|$ since f is an isometry. Moreover the points Q, M', P are collinear since P, M, Q are collinear and f is an isometry. Therefore $M=M'$.

(b) An orientation preserving isometry of $\mathbb{R}^2$ that fixes two distinct points must be identity.

TRUE: A nontrivial orientation preserving isometry of $\mathbb{R}^2$, say f, is either a translation or a rotation. A nontrivial translation does not fix any points. On the other hand, a nontrivial rotation fixes a single point. Thus f must be identity.

(c) Every isometry of $\mathbb{R}^2$, which is not a reflection, can be written as a product of two reflections.

FALSE: A product of two reflections is orientation preserving. Thus a glide reflection, which is not a reflection, cannot be written as a product of two reflections.

2. Let G_1 and G_2 be wallpaper groups and let $\phi : G_1 \to G_2$ be a group isomorphism.

(a) If $t \in G_1$ is a translation, then show that $\phi(t) \in G_2$ is a translation.

Available in the textbook.

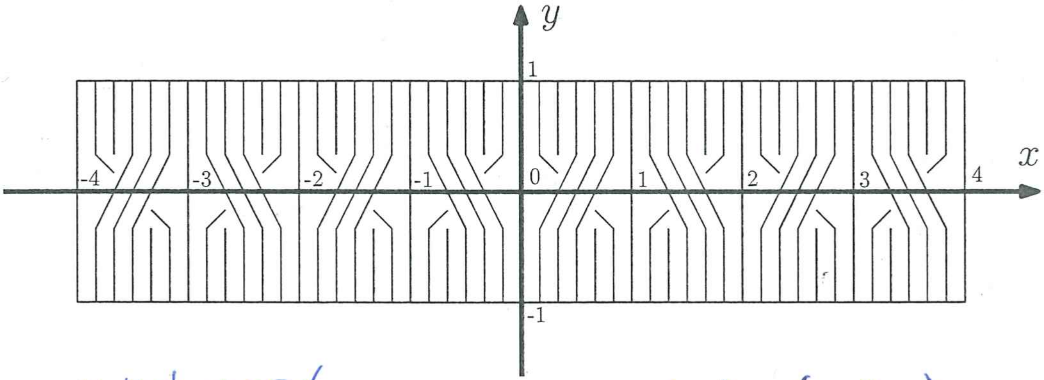
Armstrong, Groups and Geometry

Theorem 25.5, page 152.

(b) If $r \in G_1$ is a reflection, then show that $\phi(r) \in G_2$ is a reflection.

3. For each of the following, name the frieze group, determine the translation subgroup $G \cap T$ and determine the point group $J = \pi(G)$.

(a)

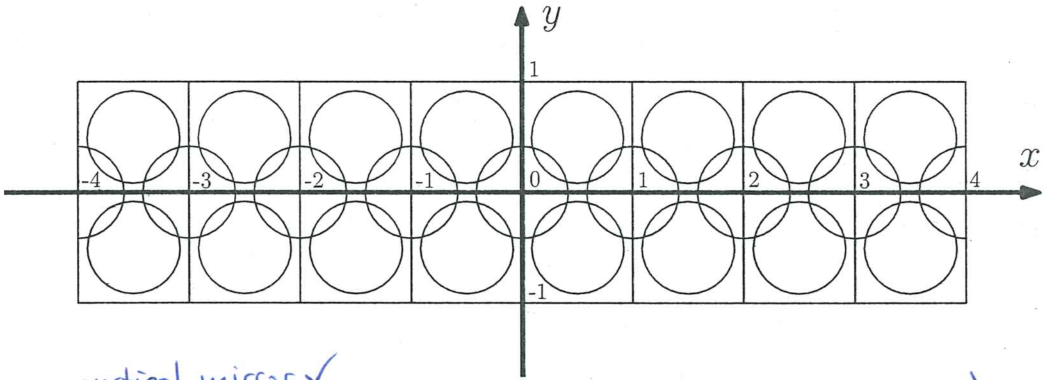


Vertical mirror ✓
horizontal glide ✓
halfturn ✓

F_2^2

Set $\tau(x,y) = (x+2, y)$
Then $G \cap T = \langle \tau \rangle$
$J = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

(b)



vertical mirror ✓
horizontal mirror ✓
halfturn ✓

F_2^1

Set $\tau(x,y) = (x+1, y)$
Then $G \cap T = \langle \tau \rangle$
$J = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right]$

Recall the following examples: F_1 or $p1$, "F F F F F F". F_1^1 or $p11m$, "B B B B B B". F_1^2 or $p1m1$, "V V V V V". F_1^3 or $p11g$, "F Ł F Ł F Ł". F_2 or $p2$, "S S S S S S S". F_2^1 or $p2mm$, "H H H H H H". F_2^2 or $p2mg$, "V Λ V Λ V Λ".

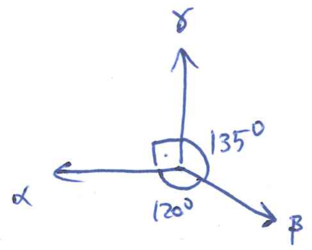
4. We set $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$, and $\gamma = e_3$ and $\mathcal{B} = \{\alpha, \beta, \gamma\}$. You are given that $W = \langle s_\alpha, s_\beta, s_\gamma \rangle$ is a finite reflection group. (This construction is referred as B_3.)

(a) Find the angels in between each pair of vectors from the basis $\mathcal{B}$.

$$\frac{\langle \alpha, \beta \rangle}{\|\alpha\| \|\beta\|} = -\frac{1}{2}$$

$$\frac{\langle \beta, \gamma \rangle}{\|\beta\| \|\gamma\|} = -\frac{1}{\sqrt{2}}$$

$$\frac{\langle \alpha, \gamma \rangle}{\|\alpha\| \|\gamma\|} = 0$$



(b) Determine the matrix representations $[s_\alpha]_{\mathcal{B}}$, $[s_\beta]_{\mathcal{B}}$ and $[s_\gamma]_{\mathcal{B}}$.

Using the formula $s_\alpha(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha$, we get

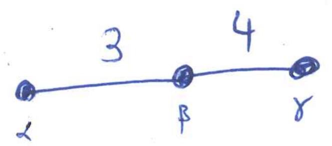
$$[s_\alpha]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[s_\beta]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[s_\gamma]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

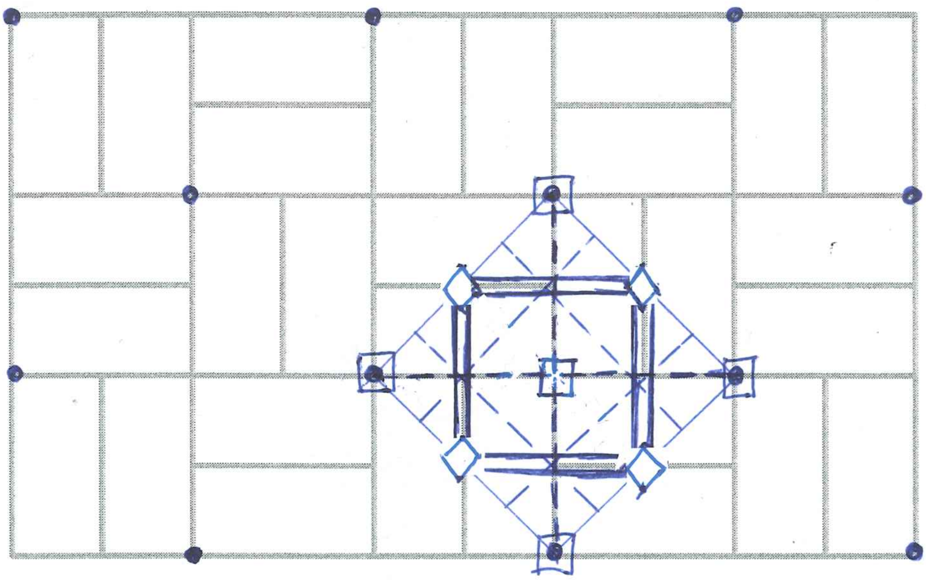
(c) Determine the Coxeter graph of B_3.

Since $\alpha \perp \gamma$, we have $m(\alpha, \gamma) = 2$. We compute that $[s_\alpha s_\beta]_{\mathcal{B}}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[s_\beta s_\gamma]^2 \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[s_\beta s_\gamma]_{\mathcal{B}}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. As a result, we find the Coxeter graph of B_3 to be.



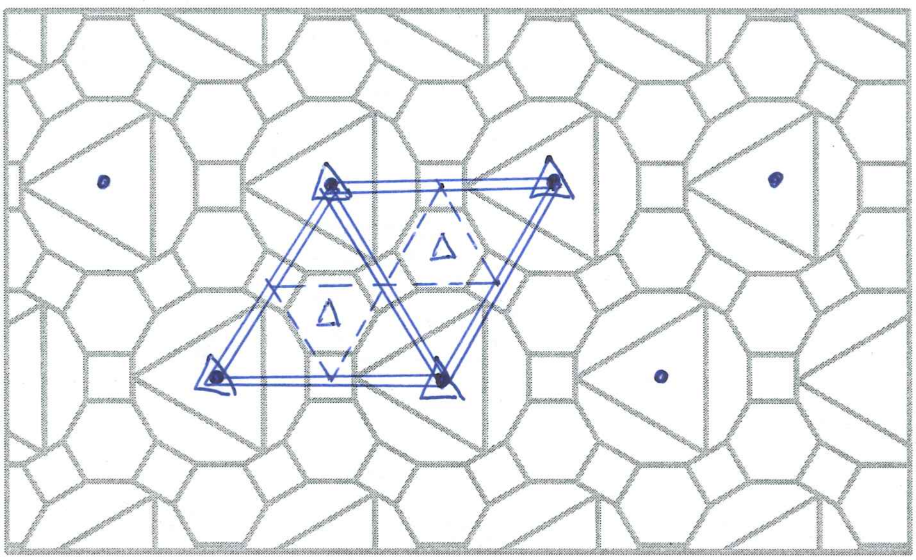
5. For each of the following, name the wallpaper group, choose a suitable lattice, put a dot at each lattice point, find a fundamental region, indicate all the centers of rotations and all the mirrors within this fundamental region.

   (a) Each rectangle is of size 1×2.



p4g

   (b) Each polygon is regular. The triangles meet the squares at the midpoint of their edges.



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• ⇝ lattice points

◇ ⇝ halfturn

△ ⇝ 3-fold rotation

□ ⇝ 4-fold rotation

═ mirror for reflection

--- axis of glide