M E T U - Department of Mathematics **Math 464 - Introduction to Representation Theory**		
Spring 2019 Ö. Küçüksakallı	Midterm 2 April 24, 17:40 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) Suppose that χ is a non-zero, non-trivial character of G. If $\chi(g)$ is a non-negative real number for all $g \in G$ then χ is reducible.

True! Let $\chi_1 = \mathbb{1}$ be the trivial character. We have

$$\langle \chi, \chi_1 \rangle = \frac{1}{|G|} \sum \chi(g) \overline{\mathbb{1}(g)} > \frac{1}{|G|} \chi(1)\overline{\chi_1(1)} > 0$$

Thus χ_1 is a constituent of χ. Moreover $\chi \neq \mathbb{1}$. Thus χ is reducible.

(b) Let χ_{reg} be the character of the regular $\mathbb{C}G$-module. If χ is a character of G, then $\langle \chi_{\text{reg}}, \chi \rangle = \chi(1)$

True! We know that $\chi_{reg} = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$

$$\langle \chi_{reg}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{reg}(g) \overline{\chi(g)} = \frac{1}{|G|} |G| \overline{\chi(1)} = \chi(1)$$

(c) Let $V = \mathbb{C}[G]$. Consider the map $\vartheta : V \to V$ given by $\vartheta : v \mapsto zv$ for some fixed $z \in V$. If ϑ is a $\mathbb{C}G$-module homomorphism then z is in the center of $\mathbb{C}[G]$.

True! Let $g \in G$. Then $\theta(gv) = g\theta(v)$ by assumption. It follows that $zgv = gzv$ for all v. Fixing $v = 1_G$ we obtain $zg = gz$ for all $g \in G$. An arbitrary element of V is a linear combination of $g \in G$. Thus $zv = vz$ for all $v \in V$ and $z \in Z(\mathbb{C}[G])$.

Question 2. (25 point) Schur's Lemma and an application.

(a) State Schur's Lemma. Give an outline for its proof.

See your textbook.
Chapter 9

(b) Let G be group and let ρ be a representation of G over $\mathbb{C}$. Show that $\rho(g)$ is diagonalizable for each $g \in G$.

See your textbook
Chapter 9

Question 3. **(25 point)** Let $G = D_8 = \langle a, b | a^4 = 1 = b^2, b^{-1}ab = a^{-1} \rangle$. Consider the following character χ of G:

g	1	a^2	a, a^3	b, a^2b	ab, a^3b
$\chi(g)$	2	2	0	-2	0

(a) Compute $\langle \chi, \chi \rangle$. Is χ irreducible?

$$\langle \chi, \chi \rangle = \frac{1}{8}\left(2 \cdot 2 + 2 \cdot 2 + 2 \cdot 0 \cdot 0 + 2 \cdot (-2)(-2) + 2 \cdot 0 \cdot 0\right)$$

$$= 2$$

The character χ is reducible since $\langle \chi, \chi \rangle = 2 \neq 1$.

(b) Show that $\chi \neq 2\psi$ for any character ψ of G.

If $\chi = 2\psi$, then $2 = \langle \chi, \chi \rangle = \langle 2\psi, 2\psi \rangle = 4 \langle \psi, \psi \rangle$.
It follows that $\langle \psi, \psi \rangle = \frac{1}{2}$, a contradiction.

(c) Find a representation ρ of G such that the associated $\mathbb{C}G$-module has character χ.

Consider $\rho : a \longmapsto A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\rho : b \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Then $A^4 = I = B^2$ and $B^{-1}AB = A^{-1}$. Thus ρ induces a group homomorphism. It is easy to verify that the associated $\mathbb{C}G$-module has character χ.

Question 4. **(25 point)** Let $G = S_3$. Recall that there are precisely three non-isomorphic $\mathbb{C}G$-modules V_1, V_2 and V_3 with characters, χ_1, χ_2 and χ_3, respectively:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

(a) Let $\mathbb{C}[G] = W_1 \oplus W_2 \oplus W_3$ where W_i is the sum of those $\mathbb{C}G$-modules which have character χ_i. Determine $e_i \in W_i$ such that $1 = e_1 + e_2 + e_3$.

In general $e_i = \dfrac{\chi_i(1)}{|G|} \sum_{g\in G} \chi(g^{-1})g$

In particular
$$e_1 = \frac{1}{6}\big((1)+(12)+(13)+(23)+(123)+(132)\big)$$
$$e_2 = \frac{1}{6}\big((1)-(12)-(13)-(23)+(123)+(132)\big)$$
$$e_3 = \frac{1}{6}\big(2(1)-(123)-(132)\big)$$

(b) Let V be the permutation module with the natural basis $\mathcal{B} = \{v_1, v_2, v_3\}$. Compute e_iV for each $i \in \{1, 2, 3\}$.

Firstly, $e_1v_1 = e_1v_2 = e_1v_3 = \frac{1}{3}(v_1+v_2+v_3)$. It follows that $e_1V = \mathrm{span}(\{v_1+v_2+v_3\})$

Secondly $e_1v_1 = e_2v_2 = e_2v_3 = 0$. Thus $e_2V = \{0\}$.

Finally
$$\left.\begin{aligned} e_3v_1 &= \tfrac{1}{3}(2v_1-v_2-v_3) \\ e_3v_2 &= \tfrac{1}{3}(2v_2-v_3-v_1) \\ e_3v_3 &= \tfrac{1}{3}(2v_3-v_1-v_2) \end{aligned}\right\} \Rightarrow e_3V = \mathrm{Span}(\{2v_1-v_2-v_3,\ 2v_2-v_3-v_1\})$$
$$= \{\lambda_1v_1+\lambda_2v_2+\lambda_3v_3 \mid \lambda_1+\lambda_2+\lambda_3=0\}$$

(c) Decompose V as a direct sum of irreduaible $\mathbb{C}G$-modules using part (b).

We have $V = e_1V \oplus e_3V = V_1 \oplus V_3$