M E T U - Department of Mathematics
Math 464 - Introduction to Representation Theory

Spring 2019 Ö. Küçüksakallı	Midterm 2 April 24, 17:40 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) Suppose that χ is a non-zero, non-trivial character of G. If $\chi(g)$ is a non-negative real number for all $g \in G$ then χ is reducible.

(b) Let χ_{reg} be the character of the regular $\mathbb{C}G$-module. If χ is a character of G, then $\langle \chi_{\text{reg}}, \chi \rangle = \chi(1)$

(c) Let $V = \mathbb{C}[G]$. Consider the map $\vartheta : V \to V$ given by $\vartheta : v \mapsto zv$ for some fixed $z \in V$. If ϑ is a $\mathbb{C}G$-module homomorphism then z is in the center of $\mathbb{C}[G]$.

Question 2. (25 point) Schur's Lemma and an application.

(a) State Schur's Lemma. Give an outline for its proof.

(b) Let G be group and let ρ be a representation of G over $\mathbb{C}$. Show that $\rho(g)$ is diagonalizable for each $g \in G$.

Question 3. (25 point) Let $G = D_8 = \langle a, b | a^4 = 1 = b^2, b^{-1}ab = a^{-1} \rangle$. Consider the following character χ of G:

g	1	a^2	a, a^3	b, a^2b	ab, a^3b
$\chi(g)$	2	2	0	-2	0

(a) Compute $\langle \chi, \chi \rangle$. Is χ irreducible?

(b) Show that $\chi \neq 2\psi$ for any character ψ of G.

(c) Find a representation ρ of G such that the associated $\mathbb{C}G$-module has character χ.

Question 4. **(25 point)** Let $G = S_3$. Recall that there are precisely three non-isomorphic $\mathbb{C}G$-modules V_1, V_2 and V_3 with characters, χ_1, χ_2 and χ_3, respectively:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

(a) Let $\mathbb{C}[G] = W_1 \oplus W_2 \oplus W_3$ where W_i is the sum of those $\mathbb{C}G$-modules which have character χ_i. Determine $e_i \in W_i$ such that $1 = e_1 + e_2 + e_3$.

(b) Let V be the permutation module with the natural basis $\mathcal{B} = \{v_1, v_2, v_3\}$. Compute e_iV for each $i \in \{1, 2, 3\}$.

(c) Decompose V as a direct sum of $\mathbb{C}G$-modules using part (b).