M E T U - Department of Mathematics
Math 464 - Introduction to Representation Theory

Spring 2019 Ö. Küçüksakallı	Midterm 1 March 20, 17:40 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) The permutation module for S_3 over $\mathbb{R}$ is faithful.

True. Recall that $ge_i = e_{g(i)}$. If $ge_i = e_i$ for each i, then $g = (1)$.

(b) Let $G = C_4 = \langle a : a^4 = 1\rangle$. There exists a nontrivial representation $\rho : G \to \mathrm{GL}(2, \mathbb{R})$ which is not faithful.

True. Consider $\rho : G \to GL(2, \mathbb{R})$ given by $a \mapsto -I_2$

(c) If V and W are $\mathbb{F}G$-modules with $\dim(V) = \dim(W)$, then V and W are isomorphic as $\mathbb{F}G$-modules.

False. Consider $G = C_2$. Let $V = \mathbb{R}$ be the trivial $\mathbb{R}G$-module given by $ax = x$ for all $x \in \mathbb{R}$. On the other hand, suppose that $W = \mathbb{R}$ be the $\mathbb{R}G$-module given by $ax = -x$ for all $x \in \mathbb{R}$. Assume $\theta : V \to W$ is an $\mathbb{F}G$-isomorphism. Then $\theta(x) = cx$ for some $c \in \mathbb{R} - \{0\}$. But $\theta(ax) = cax = -cx \neq cx = a\theta(x)$ ↯

(d) Let W be an $\mathbb{F}G$-submodule of the $\mathbb{F}G$-module V. If $V = W \oplus U$ for some subspace U of V, then U is an $\mathbb{F}G$-submodule.

False. Consider $G = C_2$ and suppose that $a(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$. Let $W = \mathrm{span}((1,1))$ and $U = \mathrm{span}((1,0))$. Now $V = W \oplus U$ but U is not an $\mathbb{F}G$-submodule.

Question 2. **(25 point)** Let $G = C_2 = \langle a : a^2 = 1 \rangle$, and let $V = \mathbb{F}^3$ with $\mathbb{F} = \mathbb{R}$ and $\mathcal{B} = \{e_1, e_2, e_3\}$. For $(x, y, z) \in V$, define

$$1(x, y, z) = (x, y, z) \quad \text{and} \quad a(x, y, z) = (y, x, -z).$$

(a) Verify that V is an $\mathbb{F}G$-module. Describe the representation given by $\rho(g) = [g]_{\mathcal{B}}$.

Consider the map $\rho : G \to GL(3, \mathbb{R})$.

$$1 \mapsto I_3$$

$$a \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The map ρ is a group homomorphism since $\rho(a)^2 = I_3$. Consider the corresponding $\mathbb{F}G$-module structure on V given by $\rho(g) = [g]_{\mathcal{B}}$. This is precisely the structure given by

$$1(x, y, z) = (x, y, z) \quad \text{and} \quad a(x, y, z) = (y, x, -z).$$

(b) Decompose V as a direct sum of irreducible FG-submodules.

Consider $w_1 = e_1 + e_2$ and $W_1 = \text{span}(w_1)$

$w_2 = e_1 - e_2$ $\quad W_2 = \text{span}(w_2)$

$w_3 = e_3$ $\quad W_3 = \text{span}(w_3)$

Note that each W_i is an $\mathbb{F}G$-submodule. Moreover, each W_i is irreducible since $\dim(W_i) = 1$. The set

$$\beta = \{w_1, w_2, w_3\}$$

is clearly a basis for V. Thus we have

$$V = W_1 \oplus W_2 \oplus W_3.$$

Question 3. (25 point) Let V be the group algebra $\mathbb{R}[G]$ with $G = S_3$.

(a) Let $x = 4(1) + (12) + (123)$ and $y = 5(12) + (13)$. Compute xy, yx and x^2.

$$xy = 20(12) + 4(13) + 5(1) + (12)(13) + 5(123)(12) + (123)(13)$$
$$= 20(12) + 4(13) + 5(1) + (132) + 5(13) + (23)$$
$$yx = 20(12) + 4(13) + 5(1) + (123) + 5(23) + (12)$$
$$x^2 = 16(1) + 4(12) + 4(123) + 4(12) + (1) + (12)(123) + 4(123) + (123)(12) + (132)$$
$$= 17(1) + 8(12) + 8(123) + (23) + (13) + (132)$$

(b) Find a **nonzero** element $z \in \mathbb{R}[G]$ such that $z((1) - (123)) = 0$.

$$z = (1) + (12) + (13) + (23) + (123) + (132)$$

or

$$z = (1) + (123) + (132)$$

(c) Let $w = (12) + (13) + (23)$. Show that $wr = rw$ for all $r \in \mathbb{R}[G]$.

It is enough to show that $gw = wg$ for each $g \in G$
The general result follows since r is a linear combination of $g \in G$.

$$1w = w1 = w$$
$$(12)w = w(12) = (1) + (123) + (132) = \alpha$$
$$(13)w = w(13) = \alpha$$
$$(23)w = w(23) = \alpha$$
$$(123)w = w(123) = w$$
$$(132)w = w(132) = w$$

This finishes the proof

Question 4. (25 point) Let $G = D_{12} = \langle a, b : a^6 = 1 = b^2, b^{-1}ab = a^{-1} \rangle$.

(a) You are given that the map $\rho : a^r b^s \mapsto A^r B^s$ is a group representation where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Is ρ is faithful? Is ρ is irreducible?

The map ρ is not faithful: $a^2 \in \mathrm{Ker}(\rho)$

The map ρ is reducible: Let V be the corresponding $\mathbb{F}G$-module with basis $\mathcal{B} = \{v_1, v_2\}$. Then $W_1 = \mathrm{span}(v_1 + v_2)$ & $W_2 = \mathrm{span}(v_1 - v_2)$ are proper $\mathbb{F}G$-submodules of V.

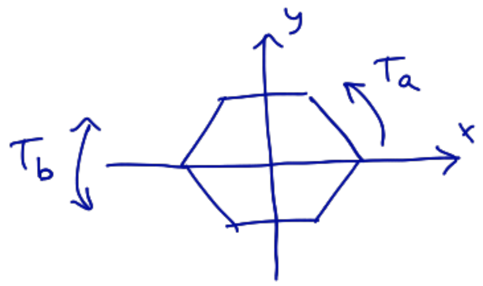
(b) You are given that the map $\sigma : a^r b^s \mapsto C^r D^s$ is a group representation where

$$C = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Is σ is faithful? Is σ is irreducible?

Let $V = \mathbb{R}^2$ be the corresponding $\mathbb{R}G$-module with basis $\mathcal{B} = \{e_1, e_2\}$. The map $T_a : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto av$ is rotation by $60°$.

Moreover the map $T_b : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto bv$ is the reflection along the x-axis. This is precisely the definition of the dihedral group with 12 elements.



Thus σ is faithful.

Now consider any nontrivial subspace of $\mathbb{R}^2$, namely a line passing through the origin. Such a line cannot remain invariant under the action of the rotation T_a. Thus $\mathbb{R}^2$ has no nontrivial submodule.