Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) If ρ is a representation of G over $\mathbb{C}$ then $\rho(g)$ is diagonalizable for each $g \in G$.

(b) If ψ is a character of G such that $\langle \psi, \psi \rangle = 4$, then there exists a character χ of G such that $\psi = 2\chi$.

(c) If G has a faithful irreducible representation over $\mathbb{C}$ then $Z(G)$ is cyclic.

Question 2. (25 point) The character table of S_5 is as follows:

	(1)	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

(a) Recall that $\chi^2 = \chi_S + \chi_A$ for certain characters χ_S and χ_A. If $\chi = \chi_6$, then determine the characters χ_S and χ_A.

	(1)	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_S							
χ_A							

(b) Write χ_6^2 as a sum of irreducible characters.

(c) Write $\chi_6^2 \chi_2$ as a sum of irreducible characters.

Question 3. (25 point) Let $G = \mathbb{Z}_5 = \langle a | a^5 = 1 \rangle$.

(a) Write the character table of G where χ_1 is used for the trivial character.

(b) Express χ_{reg} as a sum of the irreducible characters in your table.

(b) Find a character χ of G such that $\langle \chi, \chi_1 \rangle = 0$ and $\chi(g) \in \mathbb{Z}$ for all $g \in G$.

(c) Find a character ψ of G such that such that $\psi(g) \in \mathbb{R} \setminus \mathbb{Z}$ for some $g \in G$.

Question 4. (25 point) Let $G = \langle a, b | a^8 = 1 = b^2, b^{-1}ab = a^3 \rangle$ with $|G| = 16$.

(a) Show that $G' = \{1, a^2, a^4, a^6\}$. Show that G/G' is isomorphic to the Klein-4 group.

(b) Show that $Z(G) = \{1, a^4\}$. Determine $C_G(b)$ and $C_G(ab)$. Show that there are precisely 7 conjugacy classes.

(c) Fill the character table of G:

χ_1							
χ_2							
χ_3							
χ_4							
χ_5							
χ_6							
χ_7							