Name and Surname:
Student Number:

Math 366 - Spring 2017 - METU

Quiz 7

Question: Consider the ring $I_{-6} = \mathbb{Z}[\sqrt{-6}\,]$

- Is $\sqrt{-6}$ a prime element of I_{-6}?

  Solution: The element $\sqrt{-6}$ divides the product $2 \cdot 3 = 6$ in the ring I_{-6}. However it does not divide 2 or 3. We conclude that $\sqrt{-6}$ is not a prime element.

- Is $\sqrt{-6}$ an irreducible element of I_{-6}?

  Solution: Suppose that $\sqrt{-6} = \alpha\beta$. Then $6 = N(\alpha)N(\beta)$. An arbitrary element $x + y\sqrt{-6}$ of the ring I_{-6} has norm $N(x + y\sqrt{-6}\,) = x^2 + 6y^2$. This quantity is always an integer and never equals ± 2 or ± 3. As a result, either $N(\alpha) = \pm 1$ or $N(\beta) = \pm 1$. It follows that either α is a unit or β is a unit. Thus $\sqrt{-6}$ is an irreducible element.

- Does $\gcd(4 + 2\sqrt{-6}, 10)$ exist in I_{-6}?

  Solution: The set of common divisors of $4+2\sqrt{-6}$ and 10 is a nonempty partially ordered set under the divisibility relation. We note that $\alpha = 2$ and $\beta = 2 + \sqrt{-6}$ are maximal elements of this poset. However this set has no maximum element since neither $\alpha|\beta$ holds, nor $\beta|\alpha$ holds. We conclude that $\gcd(4 + 2\sqrt{-6}, 10)$ does not exist.

- Is I_{-6} a UFD?

  Solution: In a unique factorization domain, the notions *irreducible* and *prime* are equivalent. We have seen that $\sqrt{-6} \in I_{-6}$ is an irreducible element which is not prime. Thus I_{-6} is not a UFD.