Name and Surname:
Student Number:

Math 366 - Spring 2017 - METU

Quiz 6

Question: This question is related with the quadratic number field $\mathbb{Q}(\sqrt{-3})$.

- Write the norm and the trace map for $\mathbb{Q}(\sqrt{-3})$.

  Solution: If $\alpha = s + t\sqrt{-3}$ for some $s, t \in \mathbb{Q}$, then $\bar{\alpha} = s - t\sqrt{-3}$. We have

$$Tr(\alpha) = \alpha + \bar{\alpha} = 2s,$$
$$N(\alpha) = \alpha \cdot \bar{\alpha} = s^2 + 3t^2.$$

- What is I_{-3}? Describe its elements explicitly.

  Solution: We use the notation I_{-3} to indicate the set of (algebraic) integers of the field $\mathbb{Q}(\sqrt{-3})$. Since $-3 \equiv 1 \pmod 4$, we have $w_{-3} = (\sqrt{-3} + 1)/2$. Thus

$$\begin{aligned} I_{-3} &= \{x + y \cdot w_{-3} \ : \ x, y \in \mathbb{Z}\} \\ &= \left\{ \left(x + \frac{y}{2}\right) + \left(\frac{y}{2}\right)\sqrt{-3} \ : \ x, y \in \mathbb{Z} \right\}. \end{aligned}$$

- Compute $N(\alpha)$ for an arbitrary element $\alpha \in I_{-3}$.

  Solution: Combining the previous parts, for any $\alpha \in I_{-3}$ we obtain

$$N(x + y \cdot w_{-3}) = \left(x + \frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right)^2 = x^2 + xy + y^2.$$

- Is it possible to have $N(\alpha) = 5$ for some $\alpha \in I_{-3}$?

  Solution: The equation $x^2 + xy + y^2 = 5$ forms an ellipse in the xy-plane. Thus it is enough to check finitely many cases. Note that

$$\left(x + \frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right)^2 = 5.$$

  Thus $|y|$ is at most 2. Now, it is easy to verify that the equation $x^2 + xy + y^2 = 5$ has no integer solution with $-2 \leq y \leq 2$, and therefore no solution with $y \in \mathbb{Z}$.