Name and Surname:
Student Number:

Math 366 - Spring 2017 - METU

Quiz 5

Question: Fill in the blanks.

The division algorithm is an essential part of the Euclidean algorithm for $\mathbb{Z}$. Given two integers, this algorithm computes their **q**uotient and **r**emainder. More precisely, given two integers ________ a and $b \neq 0$ ________, there exist ________ integers q and r ________ such that ________ $a = bq + r$ and $|r| < |b|$ ________.

Recall that $N(\alpha) = \alpha\bar{\alpha} = x^2 + y^2$ is called the norm of a Gaussian integer $\alpha = x + yi$. We proved in class that the division algorithm works in $\mathbb{Z}[i]$, too. More precisely, given two Gaussian integers ________ α and $\beta \neq 0$ ________, there exist ________ Gaussian integers γ and δ ________ such that ________ $\alpha = \beta\gamma + \delta$ and $N(\delta) < N(\beta)$ ________.

Question: Let $R = \mathbb{Z}[\sqrt{-5}\,]$. We define the norm similarly, i.e. $N(x+y\sqrt{-5}\,) = x^2+5y^2$. Given $\alpha = 1 + \sqrt{-5} \in R$ and $\beta = 2 \in R$, show that there exists no $\gamma \in R$ (and $\delta \in R$) such that $\alpha = \gamma\beta + \delta$ and $0 \leq N(\delta) < N(\beta)$.

Solution: Assume that such an element $\gamma \in R$ (and $\delta \in R$) exists. Then the element δ must have norm less than $N(\beta) = 4$. The possibilities $N(\delta) = 2$ and $N(\delta) = 3$ are easily eliminated. If $N(\delta) = 1$, then $\delta = \pm 1$. If $N(\delta) = 0$, then $\delta = 0$. There are three cases to be considered, namely $\delta = -1, 0$ or 1.

- If $\delta = -1$, then we have $2\gamma = 2 + \sqrt{-5}$ since $\alpha = \gamma\beta + \delta$. Taking the norms of both sides, we obtain that $4N(\gamma) = 9$. This is a contradiction to the fact that $N(\gamma) \in \mathbb{Z}$.
- If $\delta = 0$, then we have $2\gamma = 1 + \sqrt{-5}$. Taking the norms of both sides, we obtain that $4N(\gamma) = 6$, another contradiction.
- If $\delta = 1$, then we have $2\gamma = \sqrt{-5}$. Taking the norms of both sides, we obtain that $4N(\gamma) = 5$, another contradiction.

We conclude that such an element $\gamma \in R$ (and $\delta \in R$) cannot exist.