Name and Surname:
Student Number:

Math 366 - Spring 2017 - METU

Quiz 3

Question: Let (a, b, c) be a Pythagorean triple such that $0 < a < b < c$. Show that there are infinitely such triples with $c - b = 1$. What can you say about such triples with $b - a = 1$?

Solution: The condition $c - b = 1$ implies that $\gcd(b, c) = 1$. Thus the Pythagorean triple (a, b, c) is primitive. Since $0 < a < b < c$, the Pythagorean triple (a, b, c) must be of the form either $(x^2 - y^2, 2xy, x^2 + y^2)$ or $(2xy, x^2 - y^2, x^2 + y^2)$ for some integers $x > y > 0$. In the latter case, both b and c are even. Thus we are left with the case $b - c = x^2 + y^2 - 2xy = 1$. This is possible only if $|x - y| = 1$ and as a result the choice $x = n + 1$ and $y = n$ for some positive integer n gives each such Pythagorean triple. More precisely $(a, b, c) = (2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1)$ with $n = 1, 2, 3, \ldots$ is an infinite family of Pythagorean triples such that $0 < a < b < c$ and $c - b = 1$.

Now, we consider the Pythagorean triples (a, b, c) with $0 < a < b < c$ and $b - a = 1$. Similar to the first case, the condition $b - a = 1$ implies that $\gcd(a, b) = 1$. Thus the Pythagorean triple (a, b, c) is primitive. Let us consider the case with $a = 2xy$ and $b = x^2 - y^2$. Then $b - a = x^2 - y^2 - 2xy = (x - y)^2 - 2y^2$. Putting $u = x - y$ and $v = y$, we obtain $u^2 - dv^2 = 1$ which is a Pell's equation with infinitely many solutions. For each solution (u, v), we consider $x = u + v$ and $y = v$. As a result, for each solution (u, v) of $u^2 - 2v^2 = 1$, the triple

$$(a, b, c) = (2(u + v)v, (u + v)^2 - v^2, (u + v)^2 + v^2)$$

is a Pythagorean triple with $b - a = 1$. For example,

u	v	a	b	c
3	2	20	21	29
17	12	696	697	985

One can show that infinitely solutions can be obtained in this fashion. There is a similar family $b - a = 1$ where a is odd. Try to construct it yourself. As a hint you may use solutions of $u^2 - 2v^2 = 1$ which are not positive. For example,

u	v	a	b	c
3	-2	-4	-3	5
17	-12	-120	-119	169