Name and Surname:
Student Number:

Math 366 - Spring 2017 - METU

Quiz 2

Question: Determine the group structure of $\text{Tor}(E(\mathbb{Q}))$ where $E : y^2 = x^3 - 2x$.

Solution: A torsion point $P(x, y)$ has a y-coordinate that equals zero or an integer dividing $D = -4(-2)^3 - 27 \cdot 0^2 = 32$ by the theorem of Nagell-Lutz. Positive values which are suitable for the y-coordinate are $0, 1, 2, 4, 8, 16$ and 32. A quick search gives the following five possible torsion points:

- $P = (-1, 1), -P = (-1, -1), Q = (0, 0), R = (2, 2)$ and $-R = (2, -2)$.

Implicit differentiation gives that $y' = (3x^2 - 2)/(2y)$. The tangent line through P is $y = m(x+1)+1$ where $m = 1/2$. The x-coordinate of $2P$ is $m^2 - (-1) - (-1) = 9/4$. Nagell-Lutz Theorem implies that $2P$ has infinite order since its coordinates are not integers. It follows that P, and therefore $-P$, has infinite order, too. The tangent line through R is $y = m(x - 2) + 2$ where $m = 5/2$. The x-coordinate of $2R$ is $(5/2)^2 - (2) - (2) = 9/4$. Similarly, the points R and $-R$ have infinite orders as well. The remaining point Q has a vertical tangent. Threfore Q has order 2. We conclude that $\text{Tor}(E(\mathbb{Q})) = \{\infty, Q\}$, a cyclic group of order 2.