METU, Spring 2017, Math 366.

Exercise Set 9

1. Find the ideal prime decomposition of (30) in I_{-29}.

2. Consider the ideals $\mathfrak{a} = (2, \sqrt{10})$ and $\mathfrak{b} = (3, 1 + \sqrt{10})$ in I_{10}. Determine if $\mathfrak{a}$ and $\mathfrak{b}$ are principal or not.

3. Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals of I_d. Show that $\mathfrak{a} + \mathfrak{b} = \gcd(\mathfrak{a}, \mathfrak{b})$ and $\mathfrak{a} \cap \mathfrak{b} = \operatorname{lcm}(\mathfrak{a}, \mathfrak{b})$.

4. Consider the ideals $\mathfrak{a} = (2 + \sqrt{-5})$ and $\mathfrak{b} = (3)$ in I_{-5}. Show that $\mathfrak{a} + \mathfrak{b} = (3, 1 - \sqrt{5})$ and $\mathfrak{a} \cap \mathfrak{b} = (9, 3 - 3\sqrt{-5})$.

5. For each of the following rings, find all ideals containing the element 30 in that ring: $\mathbb{Z}, I_{-1}, I_{-5}$.

6. For each of the following rings, find all ideals of norm 18 in that ring: I_{-1}, I_{-3}, I_3.

7. Suppose that $\mathfrak{a} = (3, 1 + \sqrt{-23})$ in I_{-23}. Show that $\mathfrak{a} \neq (1)$. Show that $N(\mathfrak{a}) = 3$. Is $\mathfrak{a}$ principal? What about $\mathfrak{a}^2$ and $\mathfrak{a}^3$.

8. Let d be a negative squarefree integer. Suppose that $\mathrm{Cl}(d)$ is trivial, i.e. I_d is a principal ideal domain. Show that $d \equiv 5 \pmod 8$ except when $d = -1, -2, -7$.

9. Show that I_d is a principal ideal domain for $-d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

10. Find the number of solutions to the Diophantine equation $x^2 + 2y^2 = 55^k$ for all positive integers k in terms of k.

11. Find the ideal prime decomposition of ideals (2) and (3) in I_7. Show that I_7 is a principal ideal domain. Does the factorization $(1 + \sqrt{7})(1 - \sqrt{7}) = (-2)(3)$ contradict to the unique factorization?

12. For each of the following justify the isomorphism: $\mathrm{Cl}(-6) \cong \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Cl}(-23) \cong \mathbb{Z}/3\mathbb{Z}$, $\mathrm{Cl}(-21) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Cl}(-39) \cong \mathbb{Z}/4\mathbb{Z}$, $\mathrm{Cl}(-103) \cong \mathbb{Z}/5\mathbb{Z}$.

13. Find two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in I_{-6} which are not principal. Show that $\mathfrak{p}_1 \sim \mathfrak{p}_2$ (without using the fact that $\mathrm{Cl}(-6) \cong \mathbb{Z}/2\mathbb{Z}$).

14. Show that the Diophantine equation $x^2 + 2015y^2 = 19^{2015}$ has no solutions.

15. Find the number of solutions of the following Diophantine equations in terms of k:
    - the equation $x^2 + 6y^2 = p^k$ for $p \in \{2, 3, 5, 7, 11, 13\}$,
    - the equation $x^2 + xy + 6y^2 = p^k$ for $p \in \{2, 3, 5, 59\}$.