METU, Spring 2017, Math 366.

Exercise Set 7

1. Let d be a squarefree integer and let $\alpha = \sqrt{d} + 1$ and $\beta = \sqrt{d} - 3$. Write $\alpha^3, \alpha\beta, \frac{\alpha+1}{\beta}$ and $\frac{\beta}{\alpha-2}$ in the form $r + s\sqrt{d}$ for some rational numbers r and s.

2. Let d_1 and d_2 be two distinct squarefree integers. Show that $\mathbb{Q}(\sqrt{d_1}) \neq \mathbb{Q}(\sqrt{d_2})$.

3. Prove the following facts about the conjugation in $\mathbb{Q}(\sqrt{d})$.
   - $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$, $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$, $\overline{\alpha/\beta} = \overline{\alpha}/\overline{\beta}$,
   - $\alpha = \overline{\alpha}$ if and only if α is rational.

4. Prove the following facts about the trace and norm maps on $\mathbb{Q}(\sqrt{d})$.
   - $Tr(\alpha + \beta) = Tr(\alpha) + Tr(\beta)$,
   - $N(\alpha\beta) = N(\alpha)N(\beta)$,
   - $N(\alpha) = 0$ if and only if $\alpha = 0$,
   - α is a root of the polynomial equation $x^2 - Tr(\alpha) + N(\alpha) = 0$.

5. Let α be a root of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree three. Consider
$$\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}.$$
Show that $\mathbb{Q}(\alpha)$ is a field. You may start with showing that α^3 is an element of $\mathbb{Q}(\alpha)$. What about $\frac{1}{\alpha-1}$?

6. Consider $\mathbb{Q}(\sqrt{5})$. Let $\alpha = 7 - 3\sqrt{5}$ and $\beta = 1 + 2\sqrt{5}$. Show that $\{\alpha, \beta\}$ is a linearly independent set over the field $\mathbb{Q}$. Express $\gamma = 2017 + 366\sqrt{5}$ as a linear combination of α and β.

7. A complex number is called an algebraic integer if it is a root of a monic irreducible polynomial with coefficients from $\mathbb{Z}$. For each of the following, determine if it is an algebraic integer or not:
$$366\sqrt{5} + 2017, \frac{\sqrt{7}+1}{2}, \frac{\sqrt[3]{19^2} + \sqrt[3]{19} + 1}{3}, \frac{\sqrt{2} + \sqrt{-1}}{2}, \frac{2\sqrt{-27} + 3}{6}.$$

8. Show that I_d, the integers of $\mathbb{Q}(\sqrt{d})$, is a subring of complex numbers. Let f be a nonzero integer. Show that $\mathbb{Z} + fI_d = \{m + f\alpha : m \in \mathbb{Z}, \alpha \in I_d\}$ is a subring of I_d. Show that $\mathbb{Z} + fI_d$ is not ideal of I_d if $|f| \geq 2$.

9. Determine the set of units in rings I_3 and I_{-3}.

10. For each of the following rings, show that it is a Euclidean domain by finding a Euclidean function: the integers $\mathbb{Z}$, the polynomial ring $\mathbb{F}[x]$ where $\mathbb{F}$ is a field, $I_{-1} = \mathbb{Z}[i], I_{-2} = \mathbb{Z}[\sqrt{-2}], I_{-3} = \mathbb{Z}[(\sqrt{-3}+1)/2]$.