METU, Spring 2017, Math 366.

Exercise Set 6

1. Show that the Gaussian integers $\mathbb{Z}[i]$ is a ring.

2. Show that $\alpha = 5 + 2i$ does not divide $\beta = 7 + 3i$. Using the Euclidean algorithm express $\gcd(\alpha, \beta) = \alpha\lambda + \beta\eta$ for some Gaussian integers λ and η.

3. Let α and β be Gaussian integers not both 0. Show that any two greatest common divisors of α and β are associates of one another.

4. Let $R = \mathbb{Z}[\sqrt{-2}]$. Let α, β be elements of R with $\beta \neq 0$. Show that there exist γ and δ such that $\alpha = \beta\gamma + \delta$ with $N(\delta) < N(\beta)$. Apply the algorithm to the specific case $\alpha = 5 + 2\sqrt{-2}$ and $\beta = 3 - \sqrt{-2}$

5. Try to prove the division algorithm for $R = \mathbb{Z}[\sqrt{-5}]$. What goes wrong? (Hint: Try to divide $\alpha = 1 + \sqrt{-5}$ by $\beta = 2$.)

6. Classify all prime elements of the ring $R = \mathbb{Z}[\sqrt{-2}]$. Is there an element $\alpha \in R$ of norm 366 or 2015?

7. Solve the equation $2x + (2 + i)y = 11 - 3i$ in the Gaussian integers $x, y \in \mathbb{Z}[i]$.

8. Let α, β be non-zero Gaussian integers. Show that $N(\gcd(\alpha, \beta)) \mid \gcd(N(\alpha), N(\beta))$.

9. Use the arithmetic of the Gaussian integers to determine all solutions to the Diophantine equation $x^2 + y^2 = z^2$. (Hint: Show that $x + iy = u\alpha^2$ for some Gaussian integer $\alpha = m + ni$ and a unit u.)

10. Let $N = p_1 p_2 \cdots p_n$ where $p_1, p_2, \ldots, p_n$ are distinct primes of the form $4k + 1$. In how many different ways can you represent N as a sum of two squares? Prove your formula by using the properties of Gaussian integers (Do not use the fact that $r_2(n)$ is a multiplicative function).

11. In how many different ways can you represent $3, 9, 27, 81, \ldots$ in the form $x^2 + 2y^2$. Do you see a pattern? Write a formula for the number of representations of 3^k in the form $x^2 + 2y^2$. Which facts do you need about the ring $\mathbb{Z}[\sqrt{-2}]$ in order to prove this formula.

12. Let a, b and c be integers such that $b^2 - 4ac = -4$ and let m be fixed positive integer. Show that the Diophantine equation $ax^2 + bxy + cy^2 = m$ has a solution if and only if the Diophantine equation $\tilde{x}^2 + \tilde{y}^2 = m$ has a solution. (Hint: A number of the form $k^2 + 1$ has no prime divisor congruent 3 modulo 4).