METU, Spring 2015, Math 366.

Exercise Set 3

1. For each of the following Diophantine equations (or system of Diophantine equations), either show that it has infinitely many nontrivial solutions or determine all solutions.

   (a) $x^2 + y^2 = z^3$.

   (b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}$.

   (c) $\frac{1}{x^4} + \frac{1}{y^4} = \frac{1}{z^4}$.

   (d) $x^2 + y^2 = z^2$ and $x^2 + z^2 = w^2$.

   (e) $x^2 + y^2 = z^2 - 1$ and $x^2 - y^2 = w^2 - 1$.

   (f) $(x^2 + y^2 - 2)^4 + 16 = z^2$.

   (g) $x^4 + y^4 = 2z^2$.

   (h) $x^4 - 4y^4 = z^2$.

2. A positive rational number n is called a *congruent number* if there is a rational right triangle with area n, i.e. if there are rational $a, b, c > 0$ such that $a^2 + b^2 = c^2$ and $ab/2 = n$. Show that 1 is not a congruent number.

3. Determine whether the following integers can be written as sums of two squares. In each case determine all possible representations as a sum of two squares.

$$n = 25, 49, 85, 125, 180, 366, 1105, 2015, 2017.$$

4. Show that any prime congruent one modulo four can be represented uniquely (aside from the order and signs of summands) as a sum of two squares.

5. If p and q are primes of the form $4k + 1$, then show that $n = p \cdot q$ can be written as a sum of two squares in at least two different ways (aside from the order and signs of summands).

6. Show that the Diophantine equation $5x^2 + 14xy + 10y^2 = n$ has a solution if and only if n is representable as a sum of two squares.

7. Show that every prime number p of the form $8k + 1$ or $8k + 3$ can be written as $p = x^2 + 2y^2$ for some integers x and y.