Name and Surname:
Student Number:

Math 366 - Spring 2015 - METU

Quiz 6

Question: Consider the ideals $\mathfrak{a} = (2, \sqrt{10})$ and $\mathfrak{b} = (3, 1+\sqrt{10})$ in I_{10}.

1. Is it true that $\mathfrak{a}^2 = (2)$? Prove or disprove.

   Solution: Yes. We have

$$\begin{aligned} \mathfrak{a}^2 &= (4, 2\sqrt{10}, 10) \\ &= (4, 2\sqrt{10}, 10, 2) \\ &= (2). \end{aligned}$$

2. Is it true that $\mathfrak{b}^2 = (3)$? Prove or disprove.

   Solution: No. We have

$$\begin{aligned} \mathfrak{b}^2 &= (9, 3(1+\sqrt{10}), 11+2\sqrt{10}) \\ &= (9, 3(1+\sqrt{10}), 11+2\sqrt{10}, 2+2\sqrt{10}) \\ &= (9, 3(1+\sqrt{10}), 11+2\sqrt{10}, 2+2\sqrt{10}, 1+\sqrt{10}) \\ &= (1+\sqrt{10}). \end{aligned}$$

   The elements 3 and $1+\sqrt{10}$ may differ by a unit. However this is not the case since $(1+\sqrt{10})/3$ is not algebraic integer.

3. Is $\mathfrak{a}\mathfrak{b}$ principal or not?

   Solution: Yes it is principal. We have

$$\begin{aligned} \mathfrak{a}\mathfrak{b} &= (6, 2(1+\sqrt{10}), 3\sqrt{10}, 10+\sqrt{10}) \\ &= (6, 2+2\sqrt{10}, 3\sqrt{10}, 10+\sqrt{10}, -2+\sqrt{10}). \end{aligned}$$

   Note that

$$\begin{aligned} 6 &= (-2+\sqrt{10})(2+\sqrt{10}), \\ 2+2\sqrt{10} &= (-2+\sqrt{10})(4+\sqrt{10}), \\ 3\sqrt{10} &= (-2+\sqrt{10})(5+\sqrt{10}), \\ 10+\sqrt{10} &= (-2+\sqrt{10})(5+2\sqrt{10}). \end{aligned}$$

   It follows that $\mathfrak{a}\mathfrak{b} = (-2+\sqrt{10})$.

   Alternative solution: Consider the elements $\gamma = -2+\sqrt{10}$ and $\gamma' = -2-\sqrt{10}$ which are of norm -6. Consider the ideals $\mathfrak{c} = (\gamma)$ and $\mathfrak{c}' = (\gamma')$ which are generated by γ and γ', respectively. The norm of the ideal $\mathfrak{c}$ is 6. The same is true for $\mathfrak{c}'$.

Thus each ideal $\mathfrak{c}$ or $\mathfrak{c}'$ must be a product of two prime ideals of norm 2 and 3. By the prime ideal decomposition theorem we know that there is only one prime ideal of norm 2 in I_{10}, namely $\mathfrak{a} = (2, \sqrt{10})$. Moreover there are two prime ideals of norm 3 in I_{10}, namely $\mathfrak{b} = (3, 1 + \sqrt{10})$ and $\mathfrak{b}' = (3, 1 - \sqrt{10})$. Now we must have $\mathfrak{ab} = \mathfrak{c}$ or $\mathfrak{c}'$ by the unique ideal prime decomposition. In either case $\mathfrak{ab}$ is principal.

In order to decide if $\mathfrak{ab} = \mathfrak{c}$ or $\mathfrak{c}'$, we can consider the following idea. Note that

$$\sqrt{-10} \equiv 2 \pmod{\mathfrak{c}}.$$

If $\mathfrak{c} = \mathfrak{ab}$ or $\mathfrak{c} = \mathfrak{ab}'$, we must have $\sqrt{-10} \equiv 2 \pmod{\mathfrak{b}}$ or $\sqrt{-10} \equiv 2 \pmod{\mathfrak{b}'}$, respectively. Note that the first congruence is true, whereas the second one is wrong. Thus we conclude that $\mathfrak{ab} = \mathfrak{c} = (-2 + \sqrt{10})$.