Name and Surname:
Student Number:

Math 366 - Spring 2015 - METU

Quiz 5

Question 1: Consider the ideals $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ in the ring $I_{-17} = \mathbb{Z}[\sqrt{-17}]$.

$$\mathfrak{a} = (2, 1 + \sqrt{-17}),$$
$$\mathfrak{b} = (3, 1 + \sqrt{-17}),$$
$$\mathfrak{c} = (3, 1 - \sqrt{-17})$$

For each of the following pairs of claims, choose one and justify. Clearly indicate the one you have chosen.

(i) $\mathfrak{a} = (2, 1 - \sqrt{-17})$ or $\mathfrak{b} \neq \mathfrak{c}$.

Solution: For the first claim, we have

$$\begin{aligned}\mathfrak{a} &= (2, 1 + \sqrt{-17}, 1 - \sqrt{-17}) \\ &= (2, 1 - \sqrt{-17}).\end{aligned}$$

For the second claim, note that $1 \in \mathfrak{b} + \mathfrak{c}$. Assume that $1 \in \mathfrak{b}$. Then $1 \in \mathfrak{c}$ by the symmetry. It follows that $1 \in \mathfrak{b}\mathfrak{c} = (3)$, a contradiction. Thus $\mathfrak{b} \neq (1)$ and there are elements in $\mathfrak{c}$ which are not contained in $\mathfrak{b}$. We conclude that $\mathfrak{b} \neq \mathfrak{c}$.

(ii) $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ or $\mathfrak{a} + \mathfrak{b} = (1)$.

Solution: For the first claim, we have $\mathfrak{a} + \mathfrak{a} = \{\alpha + \beta : \alpha, \beta \in \mathfrak{a}\} \subseteq \mathfrak{a}$. On the other hand $\mathfrak{a} = \{\alpha : \alpha \in \mathfrak{a}\} = \{\alpha + 0 : \alpha \in \mathfrak{a}\} \subseteq \mathfrak{a} + \mathfrak{a}$.

To justify the second claim note that $-2 \in \mathfrak{a}$ and $3 \in \mathfrak{b}$. Thus $1 = -2 + 3 \in \mathfrak{a} + \mathfrak{b}$. Therefore $\mathfrak{a} + \mathfrak{b} = (1)$.

(iii) $\mathfrak{a}\mathfrak{a} = (2)$ or $\mathfrak{b}\mathfrak{c} = (3)$.

Solution: To prove the first claim, observe that

$$\begin{aligned}\mathfrak{a}\mathfrak{a} &= (4, 2(1 + \sqrt{-17}), (1 + \sqrt{-17})^2) \\ &= (4, 2 + 2\sqrt{-17}, -16 + 2\sqrt{-17}, 2\sqrt{-17}) \\ &= (4, 2 + 2\sqrt{-17}, -16 + 2\sqrt{-17}, 2\sqrt{-17}, 2) \\ &= (2).\end{aligned}$$

For the second claim note that

$$\begin{aligned}\mathfrak{b}\mathfrak{c} &= (9, 3(1 + \sqrt{-17}), 3(1 - \sqrt{17}), 18) \\ &= (9, 3(1 + \sqrt{-17}), 3(1 - \sqrt{17}), 18, 6) \\ &= (9, 3(1 + \sqrt{-17}), 3(1 - \sqrt{17}), 18, 6, 3) \\ &= (3).\end{aligned}$$

(iv) $\mathfrak{ab} \neq (1+\sqrt{-17})$ or $\mathfrak{abb} = (1+\sqrt{-17})$.

Solution: Consider $\mathfrak{ab} = (6, 2(1+\sqrt{-17}), 3(1+\sqrt{-17}), (1+\sqrt{-17})^2)$. We see that $1+\sqrt{-17} \in \mathfrak{ab}$ and therefore $\mathfrak{ab} = (6, 1+\sqrt{-17})$. Assume that $\mathfrak{ab} = (1+\sqrt{-17})$. Then there exists $\alpha = x + \sqrt{-17} \in I_{-17}$ such that $6 = \alpha(1+\sqrt{-17})$. Taking the norms we find that $36 = N(\alpha) \cdot 18$. It follows that $N(\alpha) = x^2 + 17y^2 = 2$ which is a contradiction.

To justify the second claim, note that

$$\begin{aligned}
\mathfrak{abb} &= (6, 1+\sqrt{-17})(3, 1+\sqrt{-17}) \\
&= (18, 6(1+\sqrt{-17}), 3(1+\sqrt{-17}) + (1+\sqrt{-17})^2) \\
&= (18, 3(1+\sqrt{-17}), -16+2\sqrt{-17}) \\
&= (18, 3(1+\sqrt{-17}), -16+2\sqrt{-17}, 2+2\sqrt{-17}) \\
&= (18, 3(1+\sqrt{-17}), -16+2\sqrt{-17}, 2+2\sqrt{-17}, 1+\sqrt{-17}) \\
&= (1+\sqrt{-17}).
\end{aligned}$$