METU, Spring 2015, Math 366.

Exercise Set 9

1. Let $\mathfrak{p}$ be a proper ideal of a commutative ring R with identity 1_R. Show that the following are equivalent:

   - For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
   - For all ideals $\mathfrak{a}, \mathfrak{b}$ in R, $\mathfrak{ab} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

2. Find the ideal prime decomposition of (30) in I_{-29}.

3. Consider the ideals $\mathfrak{a} = (2, \sqrt{10})$ and $\mathfrak{b} = (3, 1 + \sqrt{10})$ in I_{10}. Determine if $\mathfrak{a}$ and $\mathfrak{b}$ are principal or not.

4. Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals of I_d. Show that $\mathfrak{a} + \mathfrak{b} = \gcd(\mathfrak{a}, \mathfrak{b})$ and $\mathfrak{a} \cap \mathfrak{b} = \mathrm{lcm}(\mathfrak{a}, \mathfrak{b})$.

5. Consider the ideals $\mathfrak{a} = (2 + \sqrt{-5})$ and $\mathfrak{b} = (3)$ in I_{-5}. Show that $\mathfrak{a} + \mathfrak{b} = (3, 1 - \sqrt{5})$ and $\mathfrak{a} \cap \mathfrak{b} = (9, 3 - 3\sqrt{-5})$.

6. For each of the following rings, find all ideals containing the element 30 in that ring.

   (a) $\mathbb{Z}$.

   (b) I_{-1}.

   (c) I_{-5}.

7. For each of the following rings, find all ideals of norm 18 in that ring.

   (a) I_{-1},

   (b) I_{-3}

   (c) I_3.

8. Suppose that $\mathfrak{a} = (3, 1 + \sqrt{-23})$ in I_{-23}. Show that $\mathfrak{a} \neq (1)$. Show that $N(\mathfrak{a}) = 3$. Is $\mathfrak{a}$ principal? What about $\mathfrak{a}^2$ and $\mathfrak{a}^3$.