**METU, Spring 2015, Math 366.**

# Exercise Set 8

1. Let $d$ be a squarefree negative integer. Show that the group of units $U$ of the ring of integers $I_d$ is as follows:
   - For $d = -1$, $U = \{\pm 1, \pm i\}$ where $i = \sqrt{-1}$,
   - For $d = -3$, $U = \{\pm 1, \pm\omega, \pm\omega^2\}$ where $\omega = (\sqrt{-3}+1)/2$,
   - For all other $d < 0$, $U = \{\pm 1\}$.
2. Let $R$ be an integral domain and $x, y$ are non-zero elements of $R$. Show that the following are equivalent:
   (a) $x$ and $y$ are associates, (An element $y$ is called an associate of $x$ if $x = uy$ for a unit $u$ in $R$.)
   (b) $x|y$ and $y|x$, (We write $x|y$ if there exists $z \in R$ such that $y = xz$.)
   (c) $(x) = (y)$.
3. Prove that a prime element in an integral domain is irreducible. Show that the converse is not true in general.
4. Show that $\sqrt{-5}$ is a prime element in $I_{-5}$. Is $1+\sqrt{-5}$ a prime element? What about $3+2\sqrt{-5}$?
5. Show that $I_d$ is not a UFD for $-d \in \{5, 6, 10, 13, 14, 15, 17, 21, 22, 23, 26, 29, 30\}$.
6. Is $10 = (3+i)(3-i) = 2 \cdot 5$ an example of non-unique factorization in $\mathbf{Z}[i]$.
7. Show that 6 and $2(1+\sqrt{-5})$ do not have a greatest common factor in $I_{-5}$. Do they have a least common multiple?
8. If $R$ is a Euclidean domain than the greatest common factor exist for each pair of nonzero elements $\alpha$ and $\beta$. Moreover $\gcd(\alpha,\beta)$ can be found by the Euclidean algorithm.
   (a) Show that $(\alpha,\beta) = (\gcd(\alpha,\beta))$.
   (b) Find a generator for the ideal $(x^4-1, x^3-x)$ in $\mathbf{Q}[x]$.