METU, Spring 2015, Math 366.

Exercise Set 7

1. Consider $\mathbb{Q}(\sqrt{5})$. Let $\alpha = 7 - 3\sqrt{5}$ and $\beta = 1 + 2\sqrt{5}$. Show that $\{\alpha, \beta\}$ is a linearly independent set over the field $\mathbb{Q}$. Express $\gamma = 2015 + 366\sqrt{5}$ as a linear combination of α and β.

2. A complex number is called an algebraic integer if it is a root of a monic irreducible polynomial with coefficients from $\mathbb{Z}$. For each of the following, determine if it is an algebraic integer or not.

   - $366\sqrt{5} + 2015$,
   - $(\sqrt{7} + 1)/2$,
   - $(\sqrt[3]{19^2} + \sqrt[3]{19} + 1)/3$,
   - $(\sqrt{2} + \sqrt{-1})/2$,
   - $(2\sqrt{-27} + 3)/6$

3. Show that I_d, the integers of $\mathbb{Q}(\sqrt{d})$, is a subring of complex numbers. Let f be a nonzero integer. Show that
$$\mathbb{Z} + fI_d = \{m + f\alpha : m \in \mathbb{Z}, \alpha \in I_d\}.$$
is a subring of I_d. Show that $\mathbb{Z} + fI_d$ is not ideal of I_d if $|f| \geq 2$.

4. Determine the set of units in rings I_3 and I_{-3}.

5. For each of the following rings, show that it is an Euclidean domain by finding an Euclidean function: the integers $\mathbb{Z}$, the polynomial ring $\mathbb{F}[x]$ where $\mathbb{F}$ is a field, $I_{-1} = \mathbb{Z}[i], I_{-2} = \mathbb{Z}[\sqrt{-2}], I_{-3} = \mathbb{Z}[(\sqrt{-3} + 1)/2]$.

6. Show that every Euclidean domain is a principal ideal domain. Show that every principal ideal domain is a unique factorization domain. Give an example of a unique factorization domain which is not a principal ideal domain. Give an example of a principal ideal domain which is not a Euclidean domain.

7. For each of the following ideals, determine if it is principal or not. If it is principal, find a generator.

   - the ideal $(366, 2013)$ in $\mathbb{Z}$,
   - the ideal $(9 + 7i, 4 + 7i)$ in $\mathbb{Z}[i]$,
   - the ideal $(2, 1 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}\,]$,
   - the ideal $(2, x)$ in $\mathbb{Z}[x]$.

8. Show that every maximal ideal is a prime ideal. Give an example of a non-zero prime ideal which is not maximal.