METU, Spring 2015, Math 366.

Exercise Set 5

1. Compute the convergents of the simple continued fraction $[1; 1, 2, 1, 2, 1, 2, 1, 2]$ as real numbers. Is there a special real number to which convergents approach?

2. Find the first ten terms of the inifinite continued fraction of e. Do you see a pattern?

3. For any positive integer n, find the fundamental solution of

   (a) $x^2 - (n^2 + 1)y^2 = 1$, and

   (b) $x^2 - (n^2 + 2)y^2 = 1$.

4. Find the fundamental solution of the Pell's equation $x^2 - 41y^2 = 1$.

5. Does there exist infinitely many triples satisfying the equation $a^2 + b^2 = c^2$ such that $0 < a < b < c$ and $c - b = 1$? Same question with $b - a = 1$?

6. If d is divisible by a prime $p \equiv 3 \pmod 4$, show that the negative Pell's equation $x^2 - dy^2 = -1$ has no solution.

7. Give an infinite set of solutions of the Diophantine equation $x^2 - 6y^2 = 19$.

8. Show that the Gaussian integers $\mathbb{Z}[i]$ is a ring.

9. Show that $\alpha = 5 + 2i$ does not divide $\beta = 7 + 3i$. Using the Euclidean algorithm express $\gcd(\alpha, \beta) = \alpha\lambda + \beta\eta$ for some Gaussian integers λ and η.

10. Let α and β be Gaussian integers not both 0. Show that any two greatest common divisors of α and β are associates of one another.

11. Let $R = \mathbb{Z}[\sqrt{-2}]$. Let α, β be elements of R with $\beta \neq 0$. Show that there exist γ and δ such that $\alpha = \beta\gamma + \delta$ with $N(\delta) < N(\beta)$. Apply the algorithm to the specific case $\alpha = 5 + 2\sqrt{-2}$ and $\beta = 3 - \sqrt{-2}$

12. Try to prove the division algorithm for $R = \mathbb{Z}[\sqrt{-5}]$. What goes wrong? (Hint: Try to divide $\alpha = 1 + \sqrt{-5}$ by $\beta = 2$.)

13. Classify all prime elements of the ring $R = \mathbb{Z}[\sqrt{-2}]$. Is there an element $\alpha \in R$ of norm 366 or 2015?

14. Solve the equation $2x + (2 + i)y = 11 - 3i$ in the Gaussian integers $x, y \in \mathbb{Z}[i]$.