METU, Spring 2015, Math 366.

Exercise Set 3

1. For each of the following Diophantine equations (or system of Diophantine equations), either show that it has infinitely many nontrivial solutions or determine all solutions.

   (a) $x^2 + y^2 = z^3$.

   (b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}$.

   (c) $\frac{1}{x^4} + \frac{1}{y^4} = \frac{1}{z^4}$.

   (d) $x^2 + y^2 = z^2$ and $x^2 + z^2 = w^2$.

   (e) $x^2 + y^2 = z^2 - 1$ and $x^2 - y^2 = w^2 - 1$.

   (f) $(x^2 + y^2 - 2)^4 + 16 = z^2$.

   (g) $x^4 + y^4 = 2z^2$.

   (h) $x^4 - 4y^4 = z^2$.

2. Determine whether the following integers can be written as sums of two squares. In each case determine all possible representations as a sum of two squares.

   (a) $n = 25$.

   (b) $n = 49$.

   (c) $n = 85$.

   (d) $n = 125$.

   (e) $n = 180$.

   (f) $n = 366$.

   (g) $n = 1105$

   (h) $n = 2015$.

3. Show that any prime congruent one modulo four can be represented uniquely (aside from the order and signs of summands) as a sum of two squares.

4. Show that the Diophantine equation $5x^2 + 14xy + 10y^2 = n$ has a solution if and only if n is representable as a sum of two squares.

5. Show that every prime number p of the form $8k + 1$ or $8k + 3$ can be written as $p = x^2 + 2y^2$ for some integers x and y.

6. Find the smallest integer n which can expressed as the sum of two cubes in two different ways. Is there a relation between this problem and elliptic curves? (Hint: Taxicab)