METU, Spring 2015, Math 366.

Exercise Set 10

1. Let d be a negative squarefree integer. Suppose that $\mathrm{Cl}(d)$ is trivial, i.e. I_d is a principal ideal domain. Show that $d \equiv 5 \pmod 8$ except when $d = -1, -2, -7$.

2. Show that I_d is a principal ideal domain for $-d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

3. Find the number of solutions to the Diophantine equation $x^2+2y^2 = 55^k$ for all positive integers k in terms of k.

4. Find the ideal prime decomposition of ideals (2) and (3) in I_7. Show that I_7 is a principal ideal domain. Does the factorization $(1+\sqrt{7})(1-\sqrt{7}) = (-2)(3)$ contradict to the unique factorization?

5. For each of the following justify the isomorphism.
   (a) $\mathrm{Cl}(-6) \cong \mathbb{Z}/2\mathbb{Z}$.
   (b) $\mathrm{Cl}(-23) \cong \mathbb{Z}/3\mathbb{Z}$.
   (c) $\mathrm{Cl}(-21) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
   (d) $\mathrm{Cl}(-39) \cong \mathbb{Z}/4\mathbb{Z}$.
   (e) $\mathrm{Cl}(-103) \cong \mathbb{Z}/5\mathbb{Z}$.

6. Find two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in I_{-6} which are not principal. Show that $\mathfrak{p}_1 \sim \mathfrak{p}_2$ (without using the fact that $\mathrm{Cl}(-6) \cong \mathbb{Z}/2\mathbb{Z}$).

7. Show that the Diophantine equation $x^2 + 2015y^2 = 19^{2015}$ has no solutions.

8. Find the number of solutions of the following Diophantine equations in terms of k.
   (a) $x^2 + 6y^2 = p^k$ for $p \in \{2, 3, 5, 7, 11, 13\}$.
   (b) $x^2 + xy + 6y^2 = p^k$ for $p \in \{2, 3, 5, 59\}$.