Math 366 - Spring 2013 - METU

Sample Questions

1. **Diophantine Equations**
   (a) Determine all solutions to the Diophantine equation $x^2 + y^2 = z^2$ such that $\gcd(x, y, z) = 1$.
   (b) For which positive integers n, the Diophantine equation $x^n + y^n = z^2$ has only the trivial solutions?
   (c) Describe a process which gives infinitely many primitive solutions to the system of Diophantine equations $a^2 + b^2 = c^2$ and $a - b = 17$.

2. **Gaussian Integers**
   (a) Find the greatest common divisor of the Gaussian integers $\alpha = 23 + 11\sqrt{-1}$ and $\beta = 26$. Find Gaussian integers γ and λ such that $\gcd(\alpha, \beta) = \gamma\alpha + \lambda\beta$. Find the prime factorization of α and β.
   (b) Find the number of solutions of the Diophantine equations $x^2 + y^2 = 36^6$ and $x^2 + y^2 = 20^{13}$.
   (c) Determine all solutions to the Diophantine equation $x^2 + y^2 = z^2$ such that $\gcd(x, y, z) = 1$ (using the arithmetic of Gaussian integers).

3. **Arithmetic in Quadratic Fields**
   (a) Recall that $I_d = \{\alpha \in \mathbb{Q}(\sqrt{d}) : \mathrm{Tr}(\alpha), \mathrm{N}(\alpha) \in \mathbb{Z}\}$ for some square free integer d. Show that I_d is a subring of complex numbers.
   (b) Determine the set of units in rings I_3 and I_{-3}.
   (c) Let α, β be non-zero elements of I_d where d is a square free integer. If α divides β, then show that $N(\alpha)$ divides $N(\beta)$. Show that the converse does not hold for any value of d.
   (d) Determine all solutions of the Diophantine equation $x^2 - 17y^2 = 19$.

4. **Factorization Theory in Quadratic Fields**
   (a) Show that the Diophantine equation $x^2 + 13y^2 = 73^{73}$ has no solutions.
   (b) Show that the ring $I_{-2} = \mathbb{Z}[\sqrt{-2}]$ is a PID. Show that a prime p splits in I_{-2} if and only if $p \equiv 1, 3 \pmod 8$. Find the number of solutions to the Diophantine equation $x^2 + 2y^2 = 55^m$ for all positive integers m.
   (c) Let $p \neq 2, 5$ be a prime number in $\mathbb{Z}$. Show that $(p) = \mathfrak{p}\mathfrak{p}'$ in I_{-5} if and only if $p \equiv 1, 3, 7, 9 \pmod{20}$. You are *given* that $|Cl(I_{-5})| = 2$ and $\mathfrak{p}$ is principal if and only if $p \equiv 1, 9 \pmod{20}$. Find the number of solutions to each of the Diophantine equations $x^2 + 5y^2 = m^4$ where $m \in \{7, 21, 41, 123, 2013\}$.
   (d) Show that the class group $Cl(I_{-6})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Find two distinct non-principal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subset I_{-6}$ and show that $\mathfrak{p}_1 \sim \mathfrak{p}_2$.
   (e) You are given that $Cl(I_{-14}) \cong \mathbb{Z}/4\mathbb{Z}$. Find the number of solutions to the Diophantine equation $x^2 + 14y^2 = p^{12}$ for $p \in \{7, 13, 23\}$.