Name and Surname:
Student Number:

Math 366 - Spring 2013 - METU

Quiz 6

(1) Let $w = \sqrt{-13}$. Consider the ring of integers $I_{-13} = \mathbb{Z} \oplus w\mathbb{Z}$ of the imaginary quadratic field $\mathbb{Q}(w)$.

- Is $(3, 1+w)$ a proper ideal of I_{-13}?

  Note that $3 \cdot 5 - (1+w) \cdot (1-w) = 1 \in (3, 1+w)$. Thus it is not a proper ideal.

- Is (2) a prime ideal of I_{-13}?

  Neither $1+w$, nor $1-w$ is an element of the ideal (2). However the product of these two elements is 14. Since 14 is an element of the ideal (2), it is not prime.

- Show that $\mathfrak{a} = (2, 1+w)$ is not principal.

  Assume otherwise. Then there exists $\alpha = x + yw$ in I_{-13} such that $\mathfrak{a} = (\alpha)$. It follows that $2 = \alpha\beta_1$ and $1+w = \alpha\beta_2$ for some $\beta_1, \beta_2 \in I_{-13}$. Comparing the norms we see that $N(\alpha) |\, \gcd(4, 14) = 2$. On the other hand $N(\alpha) = x^2 + 13y^2$. It follows that $N(\alpha) = 1$ and α is a unit. One can show that this is a contradiction.

- Let $\mathfrak{b} = (7, 1+w)$. Determine if the ideals $\mathfrak{a}^2, \mathfrak{b}^2, \mathfrak{a}\mathfrak{b}$ principal or not.

  It is trivial to check that $\mathfrak{a}^2 = (2)$, $\mathfrak{b}^2 = (6-w)$ and $\mathfrak{a}\mathfrak{b} = (1+w)$. Thus each one is a principal ideal. These also follow from a deeper fact that the class group of I_{-13} is of order 2 and generated by the class $[\mathfrak{a}]$ or $[\mathfrak{b}]$.

- Show that $\mathfrak{a} \sim \mathfrak{b}$. Can $\mathfrak{b}$ be principal?

  Choose $\alpha = 7$ and $\beta = 1+w$. Then $\alpha\mathfrak{a} = \beta\mathfrak{b}$. We conclude by definition that $\mathfrak{a} \sim \mathfrak{b}$. Since $\mathfrak{a}$ is not principal, $\mathfrak{b}$ is not principal either.