Name and Surname:
Student Number:

Math 366 - Spring 2013 - METU

Quiz 5

(1) Consider $I_5 = \{\alpha \in \mathbb{Q}(\sqrt{5}) : \mathrm{Tr}(\alpha), \mathrm{N}(\alpha) \in \mathbb{Z}\}$. Find an element $w \in \mathbb{Q}(\sqrt{5})$ such that $I_5 = \{a + bw : a, b \in \mathbb{Z}\}$.

Solution: Let $\alpha = r + s\sqrt{5}$ be an element of I_5 where r and s are rational numbers. From the definition of I_5, it is easy to see that $\mathrm{Tr}(\alpha) = 2r$ must be an integer. As a result there exists an integer a such that $r = a/2$. Since $\mathrm{N}(\alpha) = r^2 - 5s^2$ is also an integer, $s = b/2$ for some integer b. Observe that $\mathrm{N}(\alpha) = (a^2 - 5b^2)/4$ is an integer if and only if $a \equiv b \pmod{2}$. Thus $I_5 = \{(a + b\sqrt{5})/2 : a \equiv b \pmod{5}\}$. It is now easy to see that we can choose $w = (1 + \sqrt{5})/2$ so that the result holds.

(2) Show that $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$ are linearly independent over rational numbers. Express $\alpha = 4 + 11\sqrt{2}$ in terms of the basis $\{\alpha_1, \alpha_2\}$.

Solution: Suppose that $c_1\alpha_1 + c_2\alpha_2 = 0$ for some rational numbers c_1 and c_2. It follows that $(c_1 + c_2) \cdot 1 + (c_1 - c_2) \cdot \sqrt{2} = 0$. Since $\sqrt{2}$ is not a rational number, we must have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Therefore $c_1 = c_2 = 0$ and we conclude that α_1 and α_2 are linearly independent. Since $\{\alpha_1, \alpha_2\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, there must exist rational numbers r_1 and r_2 such that $\alpha = r_1\alpha_1 + r_2\alpha_2$. From this equality, we obtain the system of equations $r_1 + r_2 = 4$ and $r_1 - r_2 = 11$. Solving for r_1 and r_2, we find that $\alpha = \frac{15}{2}\alpha_1 - \frac{7}{2}\alpha_2$.