Name and Surname:
Student Number:

Math 366 - Spring 2013 - METU

Quiz 3

(1) You are given the following table which includes the approximate values of $ne = n \cdot \exp(1)$ for integers $1 \leq n \leq 9$ with an accuracy of 3 decimal places. Using this table, find integers p and q such that $1 \leq q \leq 10$ and $|e - p/q| < 1/(10q)$.

n	1	2	3	4	5	6	7	8	9
ne	2.718	5.436	8.154	10.873	13.591	16.309	19.027	21.746	24.464

Solution: Note that the first decimal place of entries in $m = 1$ and $n = 8$ are the same. Choose $q = n - m = 7$ and $p = [ne] - [me] = 19$. Then $|7e - 19| < 1/10$. Therefore $|e - 19/7| < 1/70$ as required.

(2) (a) What is the fundamental solution of the equation $x^2 - dy^2 = 1$. Explain why this definition makes sense. **(b)** Find the fundamental solution (x_1, y_1) of $x^2 - 6y^2 = 1$. Determine (x_3, y_3).

Solution: **(a)** The fundamental solution is the positive solution (x_1, y_1) of $x^2 - dy^2 = 1$ such that $x_1 + y_1\sqrt{d}$ is minimal. This is possible since the condition $x + y\sqrt{d} \leq M$ implies that $x \leq M$ and $y \leq M$. **(b)** Observe that $(5, 2)$ is a solution of the equation $x^2 - 6y^2 = 1$. We verify that there is no other solution with $x < 5$ or $y < 2$. Therefore the fundamental solution (x_1, y_1) is $(5, 2)$. We compute that $(5 + 2\sqrt{6})^3 = 485 + 198\sqrt{6}$. Thus $(x_3, y_3) = (485, 198)$.

(3) The system of equations $a^2 + b^2 = c^2$ and $|a - b| = 7$ has non-trivial solutions such as $(5, 12, 13), (8, 15, 17)$. Find another such solution such that c is not divisible by 7.

Solution: We want to find a primitive solution of the equation $a^2 + b^2 = c^2$ such that $|a - b| = 7$. Without loss of generality, we have $a = m^2 - n^2$ and $b = 2mn$ for some relatively prime positive integers m and n. In other words

$$|m^2 - n^2 - 2mn| = |(m - n)^2 - 2n^2| = 7.$$

Set $u = m - n$ and $v = n$ and look for solutions of the equation $u^2 - 2v^2 = 7$ or $u^2 - 2v^2 = -7$. It is easy to see that $(u, v) = (3, 1)$ and $(u, v) = (1, 2)$ are solutions of these equations respectively. In order to find other solutions we multiply $3 + \sqrt{2}$ and $1 + 2\sqrt{2}$ with the quantity $3 + 2\sqrt{2}$ which corresponds to the fundamental solution of the Pell equation $u^2 - 2v^2 = 1$.

	(u, v)	(m, n)	(a, b, c)
$(3 + \sqrt{2}) \cdot (3 + 2\sqrt{2})^0$	$(3, 1)$	$(4, 1)$	$(15, 8, 17)$
$(3 + \sqrt{2}) \cdot (3 + 2\sqrt{2})^1$	$(13, 9)$	$(22, 9)$	$(403, 396, 565)$
$(1 + 2\sqrt{2}) \cdot (3 + 2\sqrt{2})^0$	$(1, 2)$	$(3, 2)$	$(5, 12, 13)$
$(1 + 2\sqrt{2}) \cdot (3 + 2\sqrt{2})^1$	$(11, 8)$	$(19, 8)$	$(297, 304, 425)$