Name and Surname:
Student Number:

Math 366 - Spring 2013 - METU

Quiz 2

(1) Does there exist infinitely many triples satisfying the equation $a^2 + b^2 = c^2$ such that $0 < a < b < c$ and $c - b = 1$? Same question with $b - a = 1$?

Solution: Note that $a^2 = c^2 - b^2 = (c + b)(c - b)$. Choosing $c + b = \square$, square of an odd integer, and $c - b = 1$, we obtain several solutions to $a^2 + b^2 = c^2$ with required property. In order to parametrize such solutions let us set $c + b = (2n + 1)^2$ for some $n \in \mathbb{N}$. Then $c = 2n^2 + 2n + 1, b = 2n^2 + 2n$ and $a = 2n + 1$ for $n \in \mathbb{N}$.

Now let us consider the same question with condition $b - a = 1$. We have two cases, b is either even or odd. If b is even then, the parametrization $a = m^2 - n^2, b = 2mn, c = m^2 - n^2$ gives an infinite family of solutions. We want to find integers m, n such that $b - a = 2mn - m^2 + n^2 = 1$. Note that this equation becomes $u^2 - 2v^2 = 1$ under the substitution $u = n + m, v = m$. This last equation is a Pell's equation which has infinitely many solutions. These solutions are given by $(3 + 2\sqrt{2})^n = u_n + v_n\sqrt{2}$. For each (u_i, v_i) we obtain a different Pythagorean triple with consecutive legs. For example $(u_2, v_2) = (17, 12)$ and therefore $(m, n) = (12, 5)$. We find that $a = 119, b = 120, c = 169$. The other case, b odd, is similar.

(2) Consider the Diophantine equation $2(x^2+y^2) = z^2$. A quick computer search gives the table below which includes the first few solutions with $x = 1$. Note that other solutions can be obtained by multiplying each variable by a fixed integer or switching x and y.

x	y	z
1	1	2
1	7	10
1	41	58
1	239	338
1	1393	1970
1	8119	11482

(a) Write down a parametrization which gives an infinite family of solutions.

(b) Does there exist infinitely many solutions with $x = 1$? *Hint: Use the first question.*

Solution: Suppose that $\gcd(x, y, z) = 1$. It follows that x, y are odd and z is even. It is easy to see that $(1^2 + 1^2) \cdot (x^2 + y^2) = (x - y)^2 + (x + y)^2$. Consider the equation $(x - y)^2 + (x + y)^2 = z^2$. Note that $\gcd(x - y, x + y, z) = 2$. Thus we have $x - y = 2(m^2 - n^2), x + y = 4mn, z = 2(m^2 + n^2)$ for some integers m, n relatively prime. As a result $x = 2mn + m^2 - n^2, y = 2mn - m^2 + n^2, z = 2(m^2 + n^2)$ gives an infinite family of solutions to the equation $2(x^2 + y^2) = z^2$.

Now let us focus on the case with $x = 1$. This means that $2mn + m^2 - n^2 = 1$. If we set $u = m + n$ and $v = n$, then it is equivalent to solve $u^2 - 2v^2 = 1$. This is a Pell's equation with infinitely many solutions. For example using the solution $(17, 12)$, we obtain $x = 1, y = 239, z = 338$.