# M E T U
# Department of Mathematics

| Elementary Number Theory II<br>Midterm 2 | |
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| Code : *Math 366*<br>Acad. Year : *2013*<br>Semester : *Spring*<br>Instructor : *Küçüksakallı*<br><br>Date : *April 18, 2013*<br>Time : *12:40*<br>Duration : *50 minutes* | Last Name :<br>Name :<br>Student No. :<br>Signature :<br><br>4 QUESTIONS ON 2 PAGES<br>60 TOTAL POINTS |

| 1 | 2 | 3 | 4 | ***Write neatly and justify your work!*** | |
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**1. (15pts)** Let $\alpha = 5^7 \cdot (1+2i)$. What is $N(\alpha)$? Find integers $x, y \geq 0$ such that $x^2 + y^2 = N(\alpha)$. Find the number of solutions of the Diophantine equation $x^2 + y^2 = N(\alpha)$ with $0 \leq x \leq y$.

*Solution*: The norm of $\alpha$ is $(5^7)^2 \cdot 5 = 5^{15}$. It is easy to see that $(5^7)^2 + (2 \cdot 5^7)^2 = 5^{15}$. In order to find the number of all solutions of $x^2 + y^2 = N(\alpha)$ with $0 \leq x \leq y$, let us set $\pi = 1 + 2i$. All Gaussian integers with prescribed norm is given by $\beta_k = (\pi)^k (\pi')^{15-k}$ for $k \in \{0, 1, \ldots, 15\}$. Note that $\beta_k' = \beta_{15-k}$. Therefore there are 8 such solutions.

**2. (15pts)** Find the greatest common divisor of Gaussian integers $\alpha = 38 + i$ and $\beta = 85$. Find the prime factorization of $\alpha$ and $\beta$. Find Gaussian integers $\gamma$ and $\lambda$ such that $\gcd(\alpha, \beta) = \gamma\alpha + \lambda\beta$.

*Solution*: We apply the divison algorithm. It is easy to see that $\beta = 2\alpha + (9-2i)$ and $\alpha = (4+i)(9-2i)$. Therefore the greatest common divisor of $\alpha$ and $\beta$ is $9 - 2i$. Using the inverse division algorithm we find that $9 - 2i = -2\alpha + \beta$. Thus we can pick $\gamma = -2$ and $\lambda = 1$. Now we find the prime factorizations of $\alpha$ and $\beta$. Note that $N(9 - 2i) = 85$ and a little insepection shows that $9 - 2i = (4 + i)(2 - i)$. It follows that $\alpha = (4 + i)^2(2 - i)$ and $\beta = (4 + i)(4 - i)(2 + i)(2 - i)$.

**3. (15pts)** Use the arithmetic of the Gaussian integers to determine all solutions to the Diophantine equation $x^2 + y^2 = z^2$. (Hint: Show that $x + iy = u \cdot \alpha^2$ for some Gaussian integer $\alpha$ and a unit $u$.)

*Solution*: Suppose that $(x, y, z)$ is a primitive solution of $x^2 + y^2 = z^2$, i.e. $\gcd(x, y, z) = 1$. Consider the factorization $(x + iy)(x - iy) = z^2$. Note that $\gcd(x + iy, x - iy)$ divides both $2x$ and $2y$. Since $\gcd(x, y) = 1$ and $z$ is odd, we must have $\gcd(x + iy, x - iy) = 1$. A Gaussian prime $\pi$ divides $x + iy$ an even number of times. Thus $x + iy$ has the form $u\alpha^2$ for some Gaussian integer $\alpha = m + ni$. It follows that $x + iy$ is an associate of $(m^2 - n^2) + i(2mn)$. As a result $\{x, y\} = \{\pm(m^2 - n^2), \pm 2mn\}$ produces the set of all primitive solutions if $m, n$ are relatively prime and not both odd.

**4. (15pts)** Recall that $I_{-2} = \{\alpha \in \mathbb{Q}(\sqrt{-2}) : Tr(\alpha), N(\alpha) \in \mathbb{Z}\}$. Find all elements of $I_{-2}$ of norm less than 10. Determine all primes $p < 50$ such that $N(\alpha) = p$ for some $\alpha \in I_{-2}$. Do you see a pattern?

*Solution*: We know that $I_{-2} = \{x + yw : x, y \in \mathbb{Z}\}$ where $w = \sqrt{-2}$. The norm of $x + yw$ is equal to $x^2 + 2y^2$. We list all elements of $I_{-2}$ of norm less than 10.

| $N(\alpha)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0 | $\pm 1$ | $\pm w$ | $\pm 1 \pm w$ | $\pm 2$ | $\times$ | $\pm 2 \pm w$ | $\times$ | $\pm 2w$ | $\pm 3, \pm 1 \pm 2w$ |

The primes $p$ such that $N(\alpha) = p$ for some $\alpha \in I_{-2}$ are given by the set $\{2, 3, 11, 17, 19, 41, 43, \ldots\}$. We will later see that this set is exactly $\{2\} \cup \{p \equiv 1, 3 \pmod 8\}$.