M E T U
Department of Mathematics

Elementary Number Theory II	
Midterm 1	
Code : *Math 366* Acad. Year : *2013* Semester : *Spring* Instructor : *Küçüksakallı* Date : *March 21, 2013* Time : *12:40* Duration : *50 minutes*	Last Name : Name : Student No. : Signature : 5 QUESTIONS ON 2 PAGES 60 TOTAL POINTS

1	2	3	4	5		

1. (12pts) Determine all solutions of the Diophantine equation $(x^4+1)^4 + y^8 = (z^2+1)^4$.
Solution: The equation $a^4 + b^4 = c^4$ has only the trivial solutions, i.e. either $a = 0$ or $b = 0$. Since $x^4 + 1 \neq 0$ we must have $y = 0$. Note that $|x^4 + 1| = |z^2 + 1|$. As a result $\{(t, 0, \pm t^2) : t \in \mathbb{Z}\}$ is the set of all solutions.

2. (12pts) Determine all solutions of the Pell equation $x^2 - 12y^2 = 1$.
Solution: It is easy to see that $(7, 2)$ is a positive solution. We verify that $(x^2-1)/12$ for $0 < x < 7$ and $12y^2+1$ for $0 < y < 2$ are not perfect squares. It follows that $(x_1, y_1) = (7, 2)$ is the fundamental solution. Define x_n and y_n by the equation $(7+2\sqrt{12})^n = x_n + y_n\sqrt{12}$. All solutions are given by $(\pm x_n, \pm y_n)$ and the trivial solutions $(\pm 1, 0)$.

3. (12pts) Does there exist a solution of the Diophantine equation $x^2 + y^2 = z^2$ such that $x - y$ is divisible by 3 but z is not divisible by 3. Give an example or disprove.
Solution: Suppose that $x \equiv y \pmod 3$. It follows that $x^2 \equiv y^2 \pmod 3$. A perfect square is never congruent to 2 modulo 3. Thus $x^2 + y^2 \equiv 0 \pmod 3$ and z is divisible by 3. There is no solution of $x^2 + y^2 = z^2$ such that $x - y$ is divisible by 3 whereas z is not.

4. (12pts) Express $3145 = 5 \cdot 17 \cdot 37$ as a sum of two squares. Express $7 \cdot 3145$ as a sum four squares.
Solution: Consider the Gaussian integer $(2+i)(4+i)(6+i) = 36 + 43i$. Comparing the norms of both sides we obtain that $3145 = 36^2 + 43^2$. Now consider the the quaternion $(2+i+j+k)(36+43i) = 29 + 122i + 79j - 7k$. Again by comparing norms, we find that $7 \cdot 3145 = 29^2 + 122^2 + 79^2 + 7^2$.

5. (12pts) Let $k \geq 3$ be an integer. Show that the Diophantine equation $x^2 - y^2 = m^k$ is solvable for any integer m.
Solution: Note that $m^{k-1} \equiv m \pmod 2$. Thus $m^{k-1} \pm m$ is divisible by 2. Choose $x + y = m^{k-1}$ and $x - y = m$. The pair $x = (m^{k-1}+m)/2$ and $y = (m^{k-1}-m)/2$ are integers and give a solution for the above equation for each value of m.