M E T U
Department of Mathematics

Elementary Number Theory II Final								
Code : *Math 366* Acad. Year : *2013* Semester : *Spring* Instructor : *Küçüksakallı* Date : *June 3, 2013* Time : *9:30* Duration : *120 minutes*						Last Name : Name : Student No. : Signature : 6 QUESTIONS ON 4 PAGES 80 TOTAL POINTS		
1	2	3	4	5	6	***Good luck!***		

1. (8+8=16 pts) Consider the family of Diophantine equations $x^n + y^n = z^2$ for positive integers $n \geq 1$. If $n = 2$, we obtain the Pythagoras equation for which there are infinitely many solutions. If $n = 4$, then there are only the trivial solutions by Fermat's infinite descent.

- If n is odd then show that the equation $x^n + y^n = z^2$ has a non-trivial solution.
  Solution: Choose $x = y = 2$ and $z = 2^{(n+1)/2}$.

- If n is even and $n \geq 6$ then show that the equation $x^n + y^n = z^2$ has only the trivial solutions.
  Solution: Without loss of generality, we can assume that $\gcd(x, y, z) = 1$. Rewrite the above equation as

$$(x^{n/2})^2 + (y^{n/2})^2 = z^2.$$

  Using the formula for primitive solutions of the Pythagoras equation, we find that $x^{n/2} = m^2 - n^2$ and $y^{n/2} = 2mn$. Suppose that m is even and n is odd, the other case is similar. Let m_2 be the 2-part of m. Then consider the product $(xy)^{n/2} = (m^2 - n^2) \cdot 2mn$. Comparing the 2-parts of both sides, we obtain that $(2m_2)^{n/2} = 2m_2$. This is not possible unless $n = 2$ (Pythagoras equation) or $m_2 = 0$ (trivial solutions).

2. (8+8=16 pts) Consider the ring $I_3 = \mathbb{Z} \oplus w\mathbb{Z}$ where $w = \sqrt{3}$.

- Find a unit $u \in I_3$ such that the set of units I_3^* is equal to $\{\pm u^n : n \in \mathbb{Z}\}$.
  Solution: The fundamental solution $(2, 1)$ of the Pell's equation $x^2 - 3y^2 = 1$ gives rise to the unit $u = 2 + \sqrt{3}$.

- Find a process which gives infinitely many solutions to the Diophantine equation $x^2 - 3y^2 = 13$. Illustrate your method by giving a few solutions.
  Solution: Observe that $(4, 1)$ is a solution of the equation $x^2 - 3y^2 = 13$. Define

$$x_n + y_n\sqrt{3} = (4 + \sqrt{3})(2 + \sqrt{2})^n$$

  for any integer n. Then we have $x_n^2 - 3y_n^2 = 13$ for each n. For $n = 1, 2, 3$, the corresponding solutions are $(11, 6), (40, 23), (149, 86)$.

3. (8 pts) Find the number of solutions to the Diophantine equation $x^2 + y^2 = 30^6$.
Solution: We shall count the ideals $\mathfrak{a} \subseteq \mathbb{Z}[i]$ with prescribed norm. Observe that

$$\mathfrak{a} = (1 + i)^6 \cdot (3)^3 \cdot (2 + i)^i \cdot (2 - i)^{6-i}$$

where $i \in \{0, 1, 2, 3, 4, 5, 6\}$. Since there are 7 such ideals, we have 28 solutions of the Diophantine equation $x^2 + y^2 = 30^6$.

4. (4+4+8=16 pts) In the first two parts, fill in the blanks and in the last part prove the second statement.

- Let n be a positive integer. Write $n = \underline{s \cdot n_0}$ where $\underline{n_0}$ has no square factors. Then $\mathbf{n = x^2 + y^2}$ for some integers x and y if and only if the only prime factors of $\underline{n_0}$ are among the primes $\underline{\text{2 and the primes } p \equiv 1 \pmod 4}$.
- Let n be a positive integer. Write $n = \underline{s \cdot n_0}$ where $\underline{n_0}$ has no square factors. Then $\mathbf{n = x^2 + 2y^2}$ for some integers x and y if and only if the only prime factors of $\underline{n_0}$ are among the primes $\underline{\text{2 and the primes } p \equiv 1, 3 \pmod 8}$.
- Prove the second statement.
  Solution: The Minkowski's constant $M_{-2} = 4\sqrt{2}/\pi$ is less than 2. Thus $\mathbb{Z}[-2]$ is a PID since the class group is trivial. We want to find the prime ideals of this ring of norm p, a rational prime number. Note that the ideal $(\sqrt{-2})$ has norm 2. Suppose that $p \neq 2$. Using the properties of Kronecker symbol, we find that

$$\left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) = (-1)^{(p-1)/2}(-1)^{(p^2-1)/8} = (-1)^{(p-1)(p+5)/8}.$$

  It follows that -2 is a square modulo p if and only if $p \equiv 1, 3 \pmod 8$.

5. (8 pts) Show that the Diophantine equation $x^2 + 2013y^2 = 5^{2013}$ has no solutions.
Solution: We have

$$\left(\frac{-2013}{5}\right) = \left(\frac{2}{5}\right) = -1$$

It follows that $\mathfrak{p} = (5)$ is the only prime ideal of the ring I_{-2013} whose norm is divisible by 5. It is not possible to produce an ideal which has norm 5^{2013}. Therefore the Diophantine equation above has no solution.

6. (4+4+4+4=16 pts) Consider the ring $I_{-23} = \mathbb{Z} \oplus w\mathbb{Z}$ where $w = (\sqrt{-23} + 1)/2$.

- Show that $N(a + bw) = a^2 + ab + 6b^2$ for $a, b \in \mathbb{Z}$.
  Solution: Recall that $N(a + bw) = (a + bw) \cdot (a + bw')$ where $w' = (-\sqrt{-23} + 1)/2$ is the conjugate of w. The result follows easily.
- Show that $\mathfrak{p}_3 = (3, 1 + \sqrt{-23})$ is not principal.
  Solution: Assume otherwise then we must have $\alpha = a + bw$ such that $N(\alpha) = 3$. It means that $a^2 + ab + 6b^2 = 3$ for some integers a, b and this is a contradiction.
- For a positive integer n, show that $\mathfrak{p}_3^n$ is principal if and only if $3|n$.
  Solution: Note that $N(1 + 2w) = 27$. It implies that $\mathfrak{p}_3^3$ is principal and generated by $1 + 2w$ or its conjugate. Using the class group structure, we conclude that $\mathfrak{p}_3^n$ is principal if and only if $3|n$.
- Find the number of solutions to the Diophantine equation $a^2 + ab + 6b^2 = 3^k$ for each positive integer k.
  Solution: Note that $\mathfrak{p}_3\mathfrak{q}_3, \mathfrak{p}_3^3, \mathfrak{q}_3$ are principal ideals with norms $3^2, 3^3, 3^3$ respectively. Let a_k be the number of solutions of the Diophantine equation $a^2 + ab + 6b^2 = 3^k$. For the first few values, we verify that $a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 2, a_5 = 2, a_6 = 6$. Now it is easy to see that $a_{k+6} = a_k + 4$ for each positive integer k.