M E T U - Department of Mathematics Math 262 - Linear Algebra II						
Spring 2019 Ö. Küçüksakallı		Midterm 2 April 17, 17:40 100 minutes 5 questions on 4 pages.				Surname: Name: Student No: Signature:
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Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) An operator T and its adjoint T^* have the same eigenvectors.

False! Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a+4b \\ 2a+3b \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have $T(v) = 5v$

but $T^*(v) = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$

(b) Let T be an invertible linear operator on a finite dimensional inner product space. Then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

True! We have $TT^{-1} = I = T^{-1}T$. Applying the adjoint operator we obtain $(T^{-1})^* T^* = I = T^*(T^{-1})^*$. We conclude that $(T^*)^{-1} = (T^{-1})^*$.

(c) Let T be an orthogonal operator on a finite dimensional real inner product space. Then T is diagonalizable.

False! Consider $T_{\pi/2}: \mathbb{R}^2 \to \mathbb{R}^2$, the rotation by $\pi/2$ radians. We have $T_{\pi/2} T_{\pi/2}^* = I = T_{\pi/2}^* T_{\pi/2}$ but $T_{\pi/2}$ is not diagonalizable. This is because the characteristic polynomial t^2+1 does not split over $\mathbb{R}$.

(d) Let $V = \mathbb{R}^2$ with the standard inner product. The orthogonal projection from V onto $W = \text{span}(\{(1,1)\})$ is given by $T(a,b) = (a+b, a+b)$ for all $(a,b) \in V$.

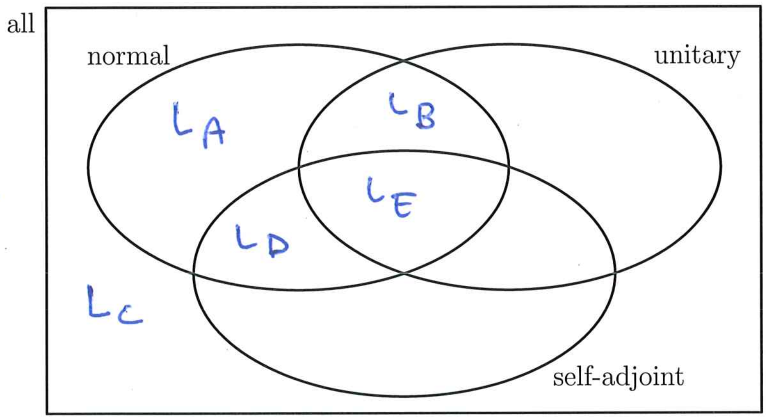
False! We have $T^2(1,1) = T(T(1,1)) = T(2,2) = (4,4)$

On the other hand $T(1,1) = (2,2)$. Since $T^2(1,1) \neq T(1,1)$, we conclude that T is not a projection, at all.

Question 2. (15 point) Let $V = \mathbb{C}^2$ be the complex vector space with the standard inner product. Recall that $L_M : V \to V$ is given by $L_M(x) = Mx$. Set

$$A = \begin{bmatrix} i & i \\ i & i \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Place the operators L_A, L_B, L_C, L_D, L_E in the correct region of the Venn diagram below.



$AA^* = A^*A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq I$

but $A \neq A^*$

$BB^* = B^*B = I$

but $B \neq B^*$

$CC^* \neq C^*C$

$DD^* = D^*D = \begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix} \neq I$

and $D = D^*$

$EE^* = E^*E = I$

and $E = E^*$

Question 3. (10 point) Let $V = P_1(\mathbb{R})$ with $\langle f, g \rangle = \int_0^2 f(t)g(t)dt$. Let $T : V \to V$ be the linear operator defined by $T(a + bx) = b$. Determine $T^*(c + dx)$.

Applying G.S. to $S = \{1, x\}$, we get $v_1 = 1$ and $v_2 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - 1$

Thus $\beta = \{\underbrace{\tfrac{1}{\sqrt{2}}}_{f_1}, \underbrace{\sqrt{\tfrac{3}{2}}(x-1)}_{f_2}\}$ is an orthonormal basis for V.

$\left(\int_0^2 1 \cdot 1 dt = 2 \text{ and } \int_0^2 (t-1)^2 dt = \frac{2}{3} \right)$

Now $[T]_\beta = \begin{bmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$ and $[T^*]_\beta = \begin{bmatrix} 0 & 0 \\ \sqrt{3} & 0 \end{bmatrix}$. We observe that

$c + dx = c + d\left(\sqrt{\tfrac{2}{3}} f_2 + 1\right) = c + d + d\sqrt{\tfrac{2}{3}} f_2 = \sqrt{2}(c+d) f_1 + \sqrt{\tfrac{2}{3}} d f_2$

So we get $T^*(c + dx) = \sqrt{3}\left(\sqrt{2}(c+d)\right) \underbrace{\sqrt{\tfrac{3}{2}}(x-1)}_{f_2} = 3(c+d)(x-1).$

Question 4. **(25 point)** Let V be a finite dimensional inner product space, and let W be a proper subspace of V.

(a) State the definition of $W^{\perp}$. Prove that $W^{\perp}$ is a subspace of V.

Defn: $W^{\perp} = \langle v \in V : \langle v, w \rangle = 0 \quad \forall w \in W \rangle$

$0 \in W^{\perp}$ because $\langle 0, w \rangle = 0 \quad \forall w \in W$

Let $c \in \mathbb{F}$ and $v_1, v_2 \in W^{\perp}$. Then $\langle v_1, w \rangle = \langle v_2, w \rangle = 0 \quad \forall w \in W$

Now $\langle cv_1 + v_2, w \rangle = c\langle v_1, w \rangle + \langle v_2, w \rangle = 0 + 0 = 0$. Thus $cv_1 + v_2 \in W^{\perp}$

(b) Let $x \in V \setminus W$. Prove that there exists $y \in W^{\perp}$ such that $\langle x, y \rangle \neq 0$.

(c) Show that $(W^{\perp})^{\perp} = W$.

Solution 1: Let $\beta = \{w_1, \dots, w_k\}$ be an orthonormal basis for W. Let us extend β to an orthonormal basis $\alpha = \{w_1, \dots, w_n\}$ for V. We write $x = \underbrace{c_1 w_1 + \dots + c_k w_k}_{z} + \underbrace{c_{k+1} w_{k+1} + \dots + c_n w_n}_{y}$. Here y is nonzero since $x \in V \setminus W$. Thus $c_i \neq 0$ for some $i \geqslant k+1$. Now

$\langle x, y \rangle = \langle y, y \rangle \geqslant |c_i|^2 > 0$

Using the preceding notation, we have $W^{\perp} = \operatorname{span}\{w_{k+1}, \dots, w_n\}$. Applying the same argument again, we find $(W^{\perp})^{\perp} = \operatorname{span}\{w_1, \dots, w_k\}$ we conclude that $W = (W^{\perp})^{\perp}$

Solution 2: Suppose that $\dim(V) = \infty$ and $\dim(W) < \infty$. In this case the first solution is not valid. There exists unique vectors $z \in W$ and $y \in W^{\perp}$ s.t. $x = z + y$. The vector y is nonzero since $x \in V \setminus W$.

Now $\langle x, y \rangle = \langle z + y, y \rangle = \underbrace{\langle z, y \rangle}_{0} + \langle y, y \rangle = \|y\| > 0$.

We have $(W^{\perp})^{\perp} \subseteq W$ because: Let $x \in (W^{\perp})^{\perp}$. Then $\langle x, y \rangle = 0 \quad \forall y \in W^{\perp}$. Assume that $x \notin W$. Then $\exists y \in W^{\perp}$ s.t. $\langle x, y \rangle \neq 0$ by the previous argument $\lightning$. Thus $x \in W$

We have $W \subseteq (W^{\perp})^{\perp}$ because: Let $x \in W$. Then $\langle y, x \rangle = 0$ for all $y \in W^{\perp}$. We have $\langle x, y \rangle = \overline{\langle y, x \rangle} = 0$ for all $y \in W^{\perp}$. It follows that $x \in (W^{\perp})^{\perp}$

Question 5. (25 point) For the matrix A below, find an orthogonal matrix Q such that $D = Q^tAQ$ is a diagonal matrix. You are given that $\det(A - tI) = -(t-2)^2(t-5)$.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

We have $\operatorname{Ker}(L_A - 2I) = \operatorname{span}\{(1,-1,0), (0,1,-1)\}$
$\operatorname{Ker}(L_A - 5I) = \operatorname{span}\{(1,1,1)\}$

Applying G.S. to $S = \{\underbrace{(1,-1,0)}_{w_1}, \underbrace{(0,1,-1)}_{w_2}\}$ we obtain

$$v_1 = w_1$$
$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (0,1,-1) - \frac{1}{2}(1,-1,0) = \left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

Eigenvectors in different eigenspaces are orthogonal to each other since A is self-adjoint. Thus

$$\beta = \left\{\frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{\sqrt{6}}(1,1,-2), \frac{1}{\sqrt{3}}(1,1,1)\right\}$$

is an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of L_A. We set

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

Now we have $QQ^t = I = Q^t \cdot Q$ so Q is an orthogonal matrix. Moreover

$$Q^tAQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$