M E T U - Department of Mathematics
Math 262 - Linear Algebra II

Spring 2019	Midterm 1	Surname:
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	100 minutes	Student No:
	4 questions on 4 pages.	Signature:

1	2	3	4		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) Let T be a linear map. The scalar zero is never an eigenvalue of T.

False. Let $T: V \to V$ be the zero map. Then any nonzero vector $v \in V$ is an eigenvector corresponding to the eigenvalue 0.

(b) Let V be a finite dimensional vector space and let W_1 be any subspace of V. Then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

True. Let β_1 be a basis of W_1. We can extend β_1 to a basis β of V. Set $\beta_2 = \beta - \beta_1$ and $W_2 = \text{Span}(\beta_2)$. Note that β is a disjoint union of β_1 and β_2.

(c) Let $A \in M_{n\times n}(\mathbb{R})$ be a square matrix. Then the dimension of $\text{span}(\{I_n, A, A^2, \ldots\})$ is less than or equal to n.

True. Let $f(t)$ be the characteristic polynomial of A. Cayley Hamilton Theorem implies that $f(A)=0$. Thus A^n is a linear combination of $I, A, \ldots, A^{n-1}$. Therefore

$$W = \text{span}(\{I_n, A, A^2, \ldots\}) \subseteq \text{span}(\{I_n, A, \ldots, A^{n-1}\})$$

Thus $\dim(W) \leq n$.

(d) The formula $\langle A, B\rangle = \text{tr}(A+B)$ defines an inner product on $M_{2\times 2}(\mathbb{R})$.

False. If $A = B = -I_2$, then $\langle A, B\rangle = \text{tr}(-2I_2) = -4$. An inner product must satisf $\langle x, x\rangle > 0$ if $x \neq 0$.

Question 2. (25 point) Let $\beta = \{1, x, x^2\}$ be the standard ordered basis for $P_2(\mathbb{R})$. Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = 2f'(x) + f(1)x$.

(a) Find $[T]_\beta$.

$$T(1) = 2\cdot 0 + 1\cdot x$$
$$T(x) = 2\cdot 1 + 1\cdot x$$
$$T(x^2) = 2\cdot 2x + 1\cdot x$$

$$[T]_\beta = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Find the characteristic polynomial of T.

$$f(t) = \det\left(\begin{bmatrix} -t & 2 & 0 \\ 1 & 1-t & 5 \\ 0 & 0 & -t \end{bmatrix}\right) = -t(t+1)(t-2)$$

(c) Find a basis for each eigenspace.

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 2$.

$$E_{\lambda_1} = \mathrm{Ker}(T + I_V) = \mathrm{Span}(\{x-2\})$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$E_{\lambda_2} = \mathrm{Ker}(T) = \mathrm{Span}(\{x^2-5\})$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

$$E_{\lambda_3} = \mathrm{Ker}(T - 2I_V) = \mathrm{Span}(\{x+1\})$$

$$\begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & 5 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(d) Find a basis α of V such that $[T]_\alpha$ is a diagonal matrix.

Choose $\alpha = \{x-2, x^2-5, x+1\}$. Then

$$[T]_\alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 3. **(25 point)** Let $V = M_{2\times 2}(\mathbb{R})$. Consider the linear map $T : V \to V$ given by the formula

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 3b & 2a+b \\ 4d-c & 5c \end{bmatrix}.$$

(a) Let W be the T-cyclic space generated by the vector $E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Find a basis for the subspace W.

Note that $E^{12} \in W$ because $T(E^{11}) = 2E^{12}$. Moreover

$$T(aE^{11}+bE^{12}) = \begin{bmatrix} 3b & 2a+b \\ 0 & 0 \end{bmatrix} = 3bE^{11} + (2a+b)E^{12}$$

Thus $W = \mathrm{Span}(\{E^{11}, E^{12}\})$. We can choose $\beta = \{E^{11}, E^{12}\}$ as a basis.

(b) Find the characteristic polynomial of the restricted map $T|_W$.

$T|_W(E^{11}) = 2E^{12}$

$T|_W(E^{12}) = 3E^{11} + E^{12}$

$$[T|_W]_\beta = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \longrightarrow f(t) = t^2 - t - 6$$

(c) Let U be the T-cyclic space generated by the vector $E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find a basis for the subspace U.

Note that $E^{21} \in U$ because $T(E^{22}) = 4E^{21}$. Moreover

$$T(cE^{21}+dE^{22}) = \begin{bmatrix} 0 & 0 \\ 4d-c & 5c \end{bmatrix} = (4d-c)E^{21} + 5cE^{22}$$

Thus $U = \mathrm{Span}(\{E^{21}, E^{22}\})$. We can choose $\gamma = \{E^{21}, E^{22}\}$ as a basis.

(d) Find the characteristic polynomial of the restricted map $T|_U$.

$T|_U(E^{21}) = -E^{21} + 5E^{22}$

$T|_U(E^{22}) = 4E^{21}$

$$[T|_U]_\gamma = \begin{bmatrix} -1 & 4 \\ 5 & 0 \end{bmatrix} \longrightarrow g(t) = t^2 + t - 20.$$

(e) Explain briefly why V is the direct sum of W and U. Find the characteristic polynomial of T **by using the parts (b) and (d).**

The standard basis $\{E^{11}, E^{12}, E^{21}, E^{22}\}$ of V is obtained as a disjoint union of β and γ. Thus $V = W \oplus U$. As a result the characteristic polynomial of T is given by

$$f(t)g(t) = (t^2 - t - 6)(t^2 + t - 20).$$

Question 4. **(25 point)** Let $V = R^3$ be the inner product space with the standard inner product.

(a) Apply the Gram-Schmidt process to $S = \{(1, 2, 2), (1, 0, 0), (0, 1, 0)\}$.

We choose $w_1 = \boxed{(1,2,2)}$, $w_2 = (1,0,0)$ and $w_3 = (0,1,0)$

Set $v_1 = w_1$. We have

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1,0,0) - \frac{1}{9}(1,2,2) = \boxed{\left(\frac{8}{9}, \frac{-2}{9}, \frac{-2}{9}\right)}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= (0,1,0) - \frac{2}{9}(1,2,2) - \frac{-2/9}{72/81}\left(\frac{8}{9}, \frac{-2}{9}, \frac{-2}{9}\right)$$

$$= (0,1,0) - \left(\frac{2}{9}, \frac{4}{9}, \frac{4}{9}\right) + \left(\frac{2}{9}, \frac{-1}{18}, \frac{-1}{18}\right)$$

$$= \boxed{\left(0, \frac{1}{2}, -\frac{1}{2}\right)}$$

Thus $S' = \left\{(1,2,2), \left(\frac{8}{9}, \frac{-2}{9}, \frac{-2}{9}\right), \left(0, \frac{1}{2}, -\frac{1}{2}\right)\right\}$ is an orthogonal set.

(b) Find an orthonormal basis β of V that contains $(1/3, 2/3, 2/3)$.

Normalizing S', we obtain

$$\beta = \Big\{ \underbrace{\tfrac{1}{3}(1,2,2)}_{u_1}, \underbrace{\tfrac{1}{3\sqrt{2}}(4,-1,-1)}_{u_2}, \underbrace{\tfrac{1}{\sqrt{2}}(0,1,-1)}_{u_3} \Big\}$$

(c) Compute the Fourier coefficients of $w = (2, 6, 2)$ relative to β. Express w as a linear combination of vectors in β.

We have $c_1 = \langle w, u_1 \rangle = \frac{1}{3}(2+12+4) = 6$

$$c_2 = \langle w, u_2 \rangle = \frac{1}{3\sqrt{2}}(8-6-2) = 0$$

$$c_3 = \langle w, u_3 \rangle = \frac{1}{\sqrt{2}}(6-2) = 2\sqrt{2}$$

Thus $w = (2,6,2) = 6u_1 + 0u_2 + 2\sqrt{2}\,u_3$