	M E T U - Department of Mathematics **Math 262 - Linear Algebra II**	

Spring 2019 Ö. Küçüksakallı	Final May 24, 13:30 120 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

Page 1	Page 2	Page 3	Page 4		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) There exists a matrix $A \in M_{3\times3}(\mathbb{C})$ such that $A^3 = I$ but $A^2 \neq I$.

TRUE! Consider the companion matrix of the polynomial t^3-1. $A=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

By construction $A^3=I$. Moreover $A^2 \neq I$.

(b) A linear operator is diagonalizable if and only if its minimal polynomial splits.

FALSE! Consider $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(f)=f'$.
The minimal polynomial of T is $p(t)=t^3$ which splits. However T is not diagonalizable.

(c) Every orthonormal set is linearly independent.

TRUE! Let S be an orthonormal set and let $v_1, \ldots, v_n \in S$.
Suppose that $a_1v_1 + \cdots + a_nv_n = 0$ for some $a_i \in \mathbb{F}$.
Now $a_i = \langle a_1v_1 + \cdots + a_nv_n, v_i \rangle = \langle 0, v_i \rangle = 0 \quad \forall i \in \{1, \ldots, n\}$.
Thus the set S is linearly independent.

(d) A generalized eigenvector of T is always an eigenvector of T.

FALSE! Consider $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(f)=f'$.
The vector $f(t)=t$ is a generalized eigenvector of T which is not an eigenvector of T.

Question 2. **(25 point)** Let $S = \{e^x, xe^x, x^2e^x, x^3e^x\}$ be a subset of real valued functions. Define $V = \text{span}(S)$. Let $T : V \to V$ be the linear operator given by $T(f) = f'$.

(a) Find the minimal polynomial of T.

Set $\alpha = S$. Then $[T]_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Characteristic polynomial of T is $f(t) = (t-1)^4$

Observe that $(T-I)(x^3e^x) = (T-I)^2(3x^2e^x)$
$= (T-I)(6xe^x)$
$= 6e^x \neq 0$

Thus the minimal polynomial of T is $p(t) = (t-1)^4$.

(b) Find a Jordan canonical basis β for V. Determine $[T]_\beta$.

Choose $g = x^3e^x \in \text{Ker}((T-I)^4) - \text{Ker}((T-I)^3)$

Set $\beta = \{(T-I)^3(g), (T-I)^2(g), (T-I)(g), g\}$
$= \{6e^x, 6xe^x, 3x^2e^x, x^3e^x\}$

Then $[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) Find a rational canonical basis γ for V. Determine $[T]_\gamma$.

Choose $g = x^3e^x \in \text{Ker}((T-I)^4) - \text{Ker}((T-I)^3)$

Set $\gamma = \{g, T(g), T^2(g), T^3(g)\}$
$= \{x^3e^x, x^3e^x + 3x^2e^x, x^3e^x + 6x^2e^x + 6xe^x, x^3e^x + 9x^2e^x + 18xe^x + 6e^x\}$

Then $[T]_\gamma = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ since $(t-1)^4 = t^4 - 4t^3 + 6t^2 - 4t + 1$

Question 3. **(25 point)** Let $V = P_2(\mathbb{R})$. Define $H(f,g) = \int_{-2}^{2} f(t)g(t)dt$.

(a) Is H a bilinear form on V? Explain briefly.

Yes! Every inner product over $\mathbb{R}$ is a bilinear form.

(b) Let $\beta = \{1, x, x^2\}$ be the standard basis for V. Find $\psi_\beta(H)$.

Note that $\int_{-2}^{2} t^n dt = \begin{cases} 2 \cdot \frac{2^{n+1}}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

$$\psi_\beta(H) = [H(x^{i-1}, x^{j-1})] = [\int_{-2}^{2} t^{i+j-2} dt]$$

$$= \begin{bmatrix} 4 & 0 & 16/3 \\ 0 & 16/3 & 0 \\ 16/3 & 0 & 64/5 \end{bmatrix}$$

(c) Find a basis γ for V such that $\psi_\gamma(H)$ is a diagonal matrix.

Performing the column and row operations $C_3 - \frac{4}{3}C_1$ and $R_3 - \frac{4}{3}R_1$, we obtain

$$\begin{bmatrix} 4 & 0 & 16/3 \\ 0 & 16/3 & 0 \\ 16/3 & 0 & 64/5 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16/3 & 0 \\ 0 & 0 & 256/45 \end{bmatrix}$$

Choosing $\gamma = \{1, x, x^2 - \frac{4}{3}\}$, we obtain

$$\psi_\beta(H) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16/3 & 0 \\ 0 & 0 & 256/45 \end{bmatrix}$$

Question 4. **(25 point)** Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find an invertible matrix Q such that $Q^{-1}AQ = B$

We have $\operatorname{Ker}(L_A - 2I) = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$

because $A - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$. Moreover $(A-2I)^2 = 0$

Choose $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \operatorname{Ker}((L_A - 2I)^2) - \operatorname{Ker}(L_A - 2I)$

Set $v_1 = (L_A - 2I)(v_2) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Choose $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in \operatorname{Ker}(L_A - 2I) - \operatorname{span}(\{v_1\})$

Now consider $\beta = \{v_1, v_2, v_3\}$. We have $[L_A]_\beta = B$.

We conclude that $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ satisfies $Q^{-1}AQ = B$.

(b) The direct approach requires 2018 matrix multiplications to compute A^{2019}. Use the previous part, to express A^{2019} as a product of at most 3 matrices.

Observe that $A^{2019} = (QBQ^{-1})^{2019} = QB^{2019}Q^{-1}$

Moreover $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$

Therefore $A^{2019} = Q \cdot \begin{bmatrix} 2^{2019} & 2019 \cdot 2^{2018} & 0 \\ 0 & 2^{2019} & 0 \\ 0 & 0 & 2^{2019} \end{bmatrix} Q^{-1}$