M E T U - Department of Mathematics
Math 261 - Linear Algebra I

Fall 2018 Ö. Küçüksakallı	Midterm 2 December 11, 17:50 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

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Question 1. (25 point) Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. **Prove** or **disprove** the following statements:

(a) If $T^2 = 0$ then $\mathrm{Im}(T) \subseteq \mathrm{Ker}(T)$.

Solution: **True.** Pick $v \in \mathrm{Im}(T)$. Then there exists $u \in V$ such that $T(u) = v$. Since $T^2 = 0$, we have $T^2(u) = T(T(u)) = T(v) = 0$. Thus $v \in \mathrm{Ker}(T)$.

(b) If $T^2 = I_V$ and $v \in V$ then there exists $w \in V$ such that $T(w) = v$.

Solution: **True.** The map is T invertible because $T \circ T = I_V$. Note that T^{-1} is the map T itself. Therefore, T is onto. This means that for each $v \in V$, there exists $w \in V$ such that $T(w) = v$. Alternatively, given $v \in V$, set $w = T(v)$. Then $v = T^2(v) = T(T(v)) = T(w)$.

(c) If $T^2 = T$ and $T \neq I_V$ then $\mathrm{Ker}(T) \neq \{0\}$.

Solution: **True.** If $T \neq I_V$ then there exists $v \in V$ such that $T(v) \neq v$. Consider the **nonzero** vector $w = T(v) - v$. We have $T(w) = T(T(v) - v) = T^2(v) - T(v)$. Using $T^2 = T$, we find that $T(w) = 0$. Thus $w \in \mathrm{Ker}(T)$. We conclude that $\mathrm{Ker}(T) \neq \{0\}$.

Question 2. (25 point) Let $\beta = \{e_1, e_2, e_3\}$ be the standard ordered basis of $V = \mathbb{R}^3$. Let $\gamma = \{u_1, u_2, u_3\}$ be another ordered basis of V with $u_1 = (1,1,1), u_2 = (1,1,0)$ and $u_3 = (0,1,1)$. Let $T : V \to V$ be the linear map such that $T(u_1) = e_1, T(u_2) = e_2$ and $T(u_3) = e_1 - e_2$.

(a) Find the rank and nullity of T. Give a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution: We have $\mathrm{Im}(T) = \mathrm{span}(\{e_1, e_2, e_1 - e_2\}) = \mathrm{span}(\{e_1, e_2\})$. The set $\{e_1, e_2\}$ is a basis for the image since it is a subset of the standard basis. We conclude that the rank of T is 2. On the other hand, the dimension theorem implies that $\mathrm{nullity}(T) = 3-2 = 1$. Observe that $T(u_1 - u_2 - u_3) = e_1 - e_2 - (e_1 - e_2) = 0$. It is easy to see that $u_1 - u_2 - u_3 = -e_2$. It follows that $\mathrm{Ker}(T) = \mathrm{span}(\{e_2\})$.

(b) Compute both $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$. Write a relation between $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$ which includes a change of coordinate matrix and its inverse.

Solution: The conditions $T(u_1) = e_1, T(u_2) = e_2$ and $T(u_3) = e_1 - e_2$ uniquely determine the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ since γ is a basis. However, these conditions are not suitable to directly compute $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$.

In order to represent T with respect to the ordered basis γ, we shall write the vectors $T(u_i)$ in terms of the vectors u_i. We observe that

$$\begin{aligned} T(u_1) &= e_1 = u_1 - u_3, \\ T(u_2) &= e_2 = -u_1 + u_2 + u_3, \\ T(u_3) &= e_1 - e_2 = 2u_1 - u_2 - 2u_3, \end{aligned} \qquad [T]_\gamma^\gamma = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}.$$

In order to represent T with respect to the standard ordered basis, we shall write the vectors $T(e_1)$ in terms of the vectors e_i. Note that

$$\begin{aligned} T(e_1) &= T(u_1 - u_3) = e_1 - (e_1 - e_2) = e_2, \\ T(e_2) &= T(-u_1 + u_2 + u_3) = \ldots = 0, \\ T(e_3) &= T(u_1 - u_2) = e_1 - e_2, \end{aligned} \qquad [T]_\beta^\beta = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The vectors u_i can be written in terms of e_i naturally. This gives us the change of coordinate matrix $Q = [I_V]_\gamma^\beta$. Its inverse, namely $Q^{-1} = [I_V]_\beta^\gamma$ can be found by expressing e_i in terms of the vectors u_i. We observe that $e_1 = u_1 - u_3, e_2 = -u_1 + u_2 + u_3$ and $e_3 = u_1 - u_2$. Now, we have

$$Q = [I_V]_\gamma^\beta = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^{-1} = [I_V]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Finally, we recall that the equality $[T]_\gamma^\gamma = Q^{-1}[T]_\beta^\beta Q$ holds. One can verify the computations above by performing the two matrix multiplications on the right hand side of this equation.

Question 3. **(25 point)** You are given that $\beta = \{(1,1,0),(2,1,0),(1,0,1)\}$ is a basis for $V = \mathbb{R}^3$. Suppose that β is an ordered basis.

(a) What is the definition of the dual space V^*? What can you say about its dimension? Does it have a natural basis?

Solution: By definition $V^* = \mathcal{L}(V, \mathbb{R})$. This is the set of all linear transformations from V to $\mathbb{R}$ and it has a vector space structure. It is of the same dimension as V, i.e. $\dim(V^*) = \dim(V) = 3$. Once a basis β is fixed for V, the set of coordinate functions β^* forms a basis for V^*.

(b) Let f_i be the ith coordinate function with respect to β. If $v = (2,6,1) \in \mathbb{R}^3$ then show that $f_1(v) = 11, f_2(v) = -5$ and $f_3(v) = 1$.

Solution: Set $v_1 = (1,1,0), v_2 = (2,1,0)$ and $v_3 = (1,0,1)$. We have $v = (2,6,1) = 11v_1 - 5v_2 + v_3$. Thus $f_1(v) = 11, f_2(v) = -5$ and $f_3(v) = 1$.

(c) Find explicit formulas for the linear functionals f_1, f_2 and f_3.

Solution: The coordinate functions satisfy the identity $f_i(v_j) = \delta_{ij}$ where δ is the Kronecker's delta function. Note that $f_1(1,1,0) = 1, f_2(1,1,0) = 0$ and $f_3(1,1,0) = 0$. Similarly, we have $f_1(2,1,0) = 0, f_2(2,1,0) = 1$ and $f_3(2,1,0)$. Finally, we have $f_1(1,0,1) = 0, f_3(1,0,1) = 0$ and $f_3(1,0,1) = 1$. Suppose that $f_i(x,y,z) = a_i x + b_i y + c_i z$ for each i. We can summarize all this information as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the columns of the matrix A consist of the vectors v_j. We need to find its inverse in order to determine the coordinate functions f_i. Applying elementary row operations $R_1 - R_3$ and $R_1 - R_2$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then we apply $R_2 - R_1$ and $R_1 \leftrightarrow R_2$ and obtain

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The inverse matrix A^{-1} includes all the coefficients a_i, b_i and c_i. Thus we have determined the coordinate functions f_i. More precisely, we have $f_1(x,y,z) = -x+2y+z$, $f_2(x,y,z) = x - y - z$, and $f_3(x,y,z) = z$.

Question 4. (25 point) Consider the linear map $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(f(x)) = (f(0), f(1), f(2)).$$

(a) Show that T is invertible and find the inverse map T^{-1}.

Solution: Consider the standard ordered bases $\beta = \{1, x, x^2\}$ and $\gamma = \{e_1, e_2, e_3\}$ for $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. We easily find that $T(1) = (1, 1, 1), T(x) = (0, 1, 2)$ and $T(x^2) = (0, 1, 4)$. Thus we have

$$A = [T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

We need to find the inverse of the matrix A. Applying the elementary row operations $R_2 - R_1, R_3 - R_1$ and then $R_3 - 2R_2$ to the augmented matrix $(A|I_3)$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

Finally, we apply $\frac{1}{2}R_3$ and $R_2 - R_3$. This gives us

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

We have found A^{-1}. Now we are ready to determine T^{-1}. Recall that $A^{-1} = [T^{-1}]_\gamma^\beta$ and $[T^{-1}(v)]_\beta = [T^{-1}]_\gamma^\beta [v]_\gamma$. If $v = (a, b, c) \in \mathbb{R}^3$ then $[v]_\gamma$ is the column vector with the same components since γ is the standard ordered basis. Therefore

$$\left[T^{-1}(v)\right]_\beta = [T^{-1}]_\gamma^\beta [v]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -\frac{3}{2}a + 2b - \frac{1}{2}c \\ \frac{1}{2}a - b + \frac{1}{2}c \end{bmatrix}$$

The last term on the right is a coordinate vector with respect to the ordered basis $\beta = \{1, x, x^2\}$. Thus we have

$$T^{-1}(a, b, c) = a \cdot 1 + \left(-\frac{3}{2}a + 2b - \frac{1}{2}c\right) x + \left(\frac{1}{2}a - b + \frac{1}{2}c\right) x^2.$$

(b) Use the computation above to find the element $T^{-1}(2, 6, 1) = g(x) \in P_2(\mathbb{R})$. Verify that $T(g(x)) = (g(0), g(1), g(2)) = (2, 6, 1)$.

Solution: It is easy to verify that $g(0) = 2, g(1) = 6$ and $g(2) = 1$ for the polynomial

$$T^{-1}(2, 6, 1) = g(x) = 2 + \frac{17}{2}x - \frac{9}{2}x^2.$$