M E T U - Department of Mathematics
Math 261 - Linear Algebra I

Fall 2018 Ö. Küçüksakallı	Midterm 1 November 6, 17:40 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 point) For each of the following statements, determine if it is true or false. Justify your answer briefly.

(a) The subset $S = \{(1,2,3),(1,1,1),(3,2,1)\}$ of $\mathbb{R}^3$ is linearly dependent.

Solution: **True.** Set $u_1 = (1,2,3), u_2 = (1,1,1)$ and $u_3 = (3,2,1)$. We have $u_1 - 4u_2 + u_3 = (0,0,0)$.

(b) Subsets of linearly independent sets are linearly independent.

Solution: **True.** This fact can be referred as the "comparison theorem" for subsets of V. Let us justify this by its contrapositive. Let S be a subset of L that is linearly dependent. Then we can find a nontrivial relation between the elements of S. The same relation holds in L. Thus L is linearly dependent.

(c) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

Solution: **False.** Consider $S = \{(1,0),(0,1),(1,1)\}$ which generates $\mathbb{R}^2$. Observe that $(1,1) = (1,0) + (0,1)$. We have expressed $(1,1)$ in two different ways as a linear combination of vectors in S.

(d) The subset $W = \{(x,y) \in \mathbb{R}^2 : x^2 = y^2\}$ is a subspace of $\mathbb{R}^2$.

Solution: **False.** The subset W is not closed under the vector addition. To see this, note that $(1,1)$ and $(1,-1)$ are in W whereas $(1,1) + (1,-1) = (2,0)$ is not in W.

(e) Let $T : V \to W$ be a linear transformation and let $S \subseteq V$ be a subset. If S is linearly dependent, then $T(S)$ is linearly dependent.

Solution: **True.** Suppose that S is linearly dependent, i.e. there exist $v_1, \ldots, v_n$ in S such that $a_1v_1 + \ldots + a_nv_n = 0$ with not all a_i are zero. Applying T, we obtain $a_1T(v_1) + \ldots + a_nT(v_n) = 0$ with not all a_i are zero. Thus $T(S)$ is linearly dependent.

Question 2. (25 point) Let $W_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$ and let W_2 be the xy-plane in $\mathbb{R}^3$.

(a) Show that W_1 is a subspace of $\mathbb{R}^3$.

Solution: It is easy to check that $1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0$. Thus $(0, 0, 0) \in W_1$. Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be elements of W_1. We have $a_1 + 2a_2 + 3a_3 = 0$ and $b_1 + 2b_2 + 3b_3 = 0$. It follows that $(a_1 + b_1) + 2(a_2 + b_2) + 3(a_3 + b_3) = 0$. Thus $(a_1, a_2, a_3) + (b_1, b_2, b_3)$ is in W_1, i.e. W_1 is closed under the vector addition. Finally we note that $ca_1 + 2ca_2 + 3ca_3 = 0$ for any real number c whenever $a_1 + 2a_2 + 3a_3 = 0$. This means that $c(a_1, a_2, a_3) \in W_1$, i.e. W_1 is closed under the scalar multiplication.

(b) Find a basis for W_1, W_2 and $W_1 \cap W_2$.

Solution: Note that $u_1 = (1, 0, -1/3)$ and $u_2 = (0, 1, -2/3)$ are elements of W_1. If $c_1u_1 + c_2u_2 = 0$, then we easily see that $c_1 = c_2 = 0$. Thus the set $\beta = \{u_1, u_2\}$ is a linearly independent subset of W_1. We want show that it generates W_1, i.e. $W_1 = \mathrm{Span}(\beta)$. Clearly, $W_1 \supseteq \mathrm{Span}(\beta)$ because W_1 is a subspace and $W_1 \supseteq \beta$. For the converse, let us pick an arbitrary element (a_1, a_2, a_3) in W_1. Then $(a_1, a_2, a_3) = (a_1, a_2, -a_1/3 - 2a_2/3) = a_1u_1 + a_2u_2$. This shows that $(a_1, a_2, a_3) \in \mathrm{Span}(\beta)$ and therefore $W_1 \subseteq \mathrm{Span}(\beta)$.

A basis for the xy-plane can be chosen to be $\gamma = \{(1, 0, 0), (0, 1, 0)\}$. It is easy to see that γ is linearly independent and that it generates the xy-plane.

Note that $W_1 \cap W_2 = \{(a_1, a_2, 0) : a_1 + 2a_2 = 0\}$. A basis for $W_1 \cap W_2$ can be chosen to be $\delta = \{(1, -1/2, 0)\}$. It is obvious that δ is linearly independent and that it generates the subspace $W_1 \cap W_2$.

(c) Show that $W_1 + W_2 = \mathbb{R}^3$.

Solution: Obviously $W_1 + W_2 \subseteq \mathbb{R}^3$. For the converse, let us pick an arbitrary element $(a_1, a_2, a_3) \in \mathbb{R}^3$. Note that $(a_1, a_2, a_3) = (-3a_3, 0, a_3) + (a_1 + 3a_3, a_2, 0)$ with $(-3a_3, 0, a_3) \in W_1$ and $(a_1 + 3a_3, a_2, 0) \in W_2$. This means that $W_1 + W_2 \supseteq \mathbb{R}^3$.

(d) Represent the zero vector of $\mathbb{R}^3$ in the form $w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$ in two different ways.

Solution: Note that $(0, 0, 0) = (0, 0, 0) + (0, 0, 0)$ trivially. As a different representation, we can consider $(0, 0, 0) = (1, -1/2, 0) + (-1, 1/2, 0)$. It is easy to see that $(1, -1/2, 0) \in W_1$ and $(-1, 1/2, 0) \in W_2$.

Question 3. (25 point) Let $V = P_{261}(\mathbb{R})$ be the vector space of polynomials with degree less than or equal to 261. Consider the subset $W = \{f(x) \in V : f(-x) = -f(x)\}$.

(a) Show that $\text{Span}(\{x, x^3, \ldots, x^{261}\}) \subseteq W$.

Solution: We start with showing that W is a subspace of V. The zero polynomial $\mathbf{0}(x)$ is in W since $\mathbf{0}(-x) \equiv 0 \equiv -\mathbf{0}(x)$. If $f(x)$ and $g(x)$ are in W, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$ by the definition of W. Now $(f+g)(-x) = f(-x) + g(-x) = -(f(x) + g(x)) = -(f+g)(x)$. We conclude that W is closed under the vector addition. Moreover $(cf)(-x) = cf(-x) = -cf(x)$. This means that W is closed under the scalar multiplication as well. Thus W is a subspace of V.

We note that $x, x^3, \ldots, x^{261}$ are elements of W since each one of these polynomials satisfy the property $f(-x) = -f(x)$. We have $\{x, x^3, \ldots, x^{261}\} \subseteq W$ and W is subspace. This proves the fact that $\text{Span}(\{x, x^3, \ldots, x^{261}\}) \subseteq W$.

(b) Show that $\text{Span}(\{x, x^3, \ldots, x^{261}\}) \supseteq W$.

Solution: Let $f(x)$ be an element of W. Since $f(x)$ is an element of V, for some real numbers a_i, we have

$$f(x) = a_{261}x^{261} + a_{260}x^{260} \ldots + a_1 x + a_0.$$

We also have $f(-x) = -f(x)$. It follows that $f(-x) + f(x) = 0$. On the other hand

$$f(-x) + f(x) = 2a_{260}x^{260} + 2a_{258}x^{258} + \ldots + 2a_2x^2 + 2a_0.$$

From this computation, we see that $a_{2i} = 0$ for each possible i. It follows that

$$f(x) = a_{261}x^{261} + a_{259}x^{259} \ldots + a_3x^3 + a_1x.$$

Now it is clear that $f(x)$ is an element of $\text{Span}(\{x, x^3, \ldots, x^{261}\})$.

(c) Is W a subspace of V? If yes, then what is the dimension of W?

Solution: We have shown that W is a subspace in the first part (a). Being a subset of the standard basis for V, the subset $\{x, x^3, \ldots, x^{261}\}$ is linearly independent. Moreover it generates the subspace W by the previous parts (a) and (b). Thus the dimension of W is the number of elements in $\{x, x^3, \ldots, x^{261}\}$, that is 131.

Question 4. (25 point) Consider the map $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$ given by the formula

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, b+c, c+d).$$

(a) Show that T is linear.

Solution: We need to show $T(kA_1 + A_2) = kT(A_1) + T(A_2)$ for arbitrary A_1 and $A_2 \in M_{2\times 2}(\mathbb{R})$ and $k \in \mathbb{R}$. Set

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

We have

$$\begin{aligned} T(kA_1 + A_2) &= T\left(\begin{bmatrix} ka_1 + a_2 & kb_1 + b_2 \\ kc_1 + c_2 & kd_1 + d_2 \end{bmatrix}\right) \\ &= (ka_1 + a_2 + kb_1 + b_2, kb_1 + b_2 + kc_1 + c_2, kc_1 + c_2 + kd_1 + d_2) \\ &= k(a_1 + b_1, b_1 + c_1, c_1 + d_1) + (a_2 + b_2, b_2 + c_2, c_2 + d_2) \\ &= kT(A_1) + T(A_2). \end{aligned}$$

(b) Find generating sets for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution: Suppose that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, b+c, c+d) = (0,0,0).$$

It follows that $a = -b = c = -d$. From this computation, we see that any element in $\mathrm{Ker}(T)$ is a scalar multiple of

$$A_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

In other words $\mathrm{Ker}(T) = \mathrm{Span}(\{A_0\})$.

The standard basis for $M_{2\times 2}(\mathbb{R})$ is $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$. The image of T is generated by $T(\beta) = \{T(E^{11}), T(E^{12}), T(E^{21}), T(E^{22})\}$. More precisely, we have

$$\mathrm{Im}(T) = \mathrm{Span}\left(\{(1,0,0), (1,1,0), (0,1,1), (0,0,1)\}\right).$$

(c) Compute the dimensions of $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution: By part (b), we have seen that $\mathrm{Ker}(T)$ is spanned by a single nonzero vector A_0. It follows that the dimension of $\mathrm{Ker}(T)$ is one. By definition, the nullity of T is one. Dimension Theorem implies that the rank of T is equal to $3 = 4 - 1$. Thus the dimension of $\mathrm{Im}(T)$ is three.