M E T U - Department of Mathematics
Math 261 - Linear Algebra I

Fall 2018	Final	Surname:
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	120 minutes	Student No:
	5 questions on 4 pages.	Signature:

1	2	3	4	5		Total

Question 1. (25 point) For each of the following statements, determine whether it is **true** or **false**. Justify your answer briefly.

(a) Consider $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = y \text{ or } y = z\}$. The set W is a subspace of $\mathbb{R}^3$ and $\beta = \{(1, 1, 0), (0, 1, 1)\}$ is a basis for W.

Solution: **False.** The set W is not a subspace of $\mathbb{R}^3$ because $(1, 1, 0) + (0, 1, 1) = (1, 2, 1)$ is not an element of W.

(b) Let V and W be vector spaces. Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Solution: **False.** Pick $x_1 = 0 \in V$ and pick a nonzero vector $y_1 \in W$. Then there exist no linear transformation T such that $T(x_1) = y_1$.

(c) For each $A \in M_{n\times n}(\mathbb{R})$ and $k \in \mathbb{R}$, we have $\det(kA) = k \det(A)$.

Solution: **False.** If $A = I_n$, then $\det(kI_n) = k^n \det(I_n)$. The above equality is false if $k \neq 0$ and $n \geq 2$.

(d) If A is an invertible matrix such that $A^{11} = A^{44}$, then $\det(A) = 1$.

Solution: **True.** If A is invertible, then $\det(A)$ is nonzero. Moreover $\det(A^n) = \det(A)^n$ for each positive integer n. Thus $\det(A)^{33} = 1$. It follows that $\det(A) = 1$ since $\det(A)$ is a real number.

(e) The system $\left\{ \begin{array}{rcl} x_1 + 2x_2 + 3x_3 & = & 2 \\ 3x_1 + 2x_2 + x_3 & = & 6 \\ x_1 + x_2 + x_3 & = & 1 \end{array} \right\}$ has no solutions.

Solution: **True.** Applying the elementary row operations $R_3 - \frac{1}{4}R_1$ and $R_3 - \frac{1}{4}R_2$, we find that the following matrices are row equivalent

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We see that $\text{rank}([A|b]) = 3$ whereas $\text{rank}(A) = 2$. Thus the system is inconsistent and it has no solutions.

Question 2. **(25 point)** Consider $W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 - 2a_2 + 3a_3 - 4a_4 = 0\}$ and $U = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 = a_2\}$. You are given that W and U are subspaces of the vector space $V = \mathbb{R}^4$.

(a) Find a basis β for W that contains $S = \{(1, 0, 1, 1), (3, 1, 1, 1)\}$.

Solution: A basis for W can be found by $Ax = 0$ with $A = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix}$. A basis for the solutions space, namely W, is given by $\beta = \{(2, 1, 0, 0), (-3, 0, 1, 0), (4, 0, 0, 1)\}$. The union $S \cup \beta$ is a generating set for W which is **not** linearly dependent. Using the Gaussian Elimination, we obtain

$$\begin{bmatrix} 1 & 3 & 2 & -3 & 4 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 & 1 \\ 0 & \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We shall remove the third and the fifth vectors from this union since they can be expressed as a linear combination of the other three vectors. Thus $\gamma = S \cup \{(-3, 0, 1, 0)\}$ is a basis for W containing S.

(b) Show that $\dim(W \cap U) = 2$ by finding a basis for $W \cap U$.

Solution: We shall solve the system of equations $a_1 - 2a_2 + 3a_3 - 4a_4 = 0$ and $a_1 - a_2 = 0$. Using the Gaussian Elimination, we obtain

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & -3 & 4 \end{bmatrix}.$$

From this computation, we find that $\alpha = \{(3, 3, 1, 0), (-4, -4, 0, 1)\}$ is a basis for $W \cap U$.

(c) Find f in the dual space of V such that $\mathrm{Ker}(f) = U$.

Solution: Consider the map $f : \mathbb{R}^4 \to \mathbb{R}$ given by the formula $f(a_1, a_2, a_3, a_4) = a_1 - a_2$. The map f is clearly linear and it satisfies $\mathrm{Ker}(f) = U$.

(d) Does there exist g in the dual space of V such that $\mathrm{Ker}(g) = W \cap U$?

Solution: Such a $g : \mathbb{R}^4 \to \mathbb{R}$ does not exist. To see this, we can use the Dimension Theorem. The nullity of g is precisely two by the part (b). Moreover the rank of g is at most 1. The rank and nullity do not add up to 4, a contradiciton.

Question 3. (25 point) Consider the linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by the formula

$$T(a, b, c, d) = (a + c + 2d\ ,\ b + 2c + 3d\ ,\ -a + b + c + d\ ,\ a - b - c).$$

(a) Find the rank and the nullity of T.

Solution: Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the standard ordered basis for $\mathbb{R}^4$. Using the Gaussian Elimination, we obtain

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this computation, we find that the nullity of T is one. Indeed, the kernel of T is spanned by $(-1, -2, 1, 0)$. By the Dimension Theorem, the rank of T must be three.

(b) Determine whether $u = (2, 0, 1, 9) \in \text{Im}(T)$ or not. Find the inverse image $T^{-1}(u)$.

Solution: Applying the elementary row operations $R_3 + R_1$ and $R_3 - R_2$, we find that

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 & 0 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{3} \\ 1 & -1 & -1 & 0 & 9 \end{array}\right]$$

The system is inconsistent because of the nonzero entry 3 in the fifth column. It follows that the inverse image $T^{-1}(u)$ is the empty set.

(c) Determine whether $v = (3, 4, 1, 0) \in \text{Im}(T)$ or not. Find the inverse image $T^{-1}(v)$.

Solution: Using the Gaussian Elimination, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In this case, the system is consistent since the rank of $[T]_\beta^\beta$ and the augmented matrix above agree. A solution to the system is easily seen to be $s = (1, 1, 0, 1)$. In other words $T(1, 1, 0, 1) = (3, 4, 0, 1)$. This is not the only element of $T^{-1}(3, 4, 1, 0)$. Recall that the solutions of the homogeneous system is spanned by $(-1, -2, 1, 0)$. Thus

$$T^{-1}(3, 4, 1, 0) = \{(1 - t, 1 - 2t, t, 1) \mid t \in \mathbb{R}\}.$$

Question 4. (10 point) Let V and W be finite dimensional vector spaces such that $\dim(\mathrm{V}) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Solution: Set $n = \dim(\mathrm{V}) = \dim(W)$. Let $B = \{v_1, \ldots, v_k\}$ be a basis for $\mathrm{Ker}(T)$. We extend the set B to a basis $\beta = \{v_1 \ldots v_k, v_{k+1}, \ldots, v_n\}$ of V. Set $w_{k+1} = T(v_{k+1}), \ldots, w_n = T(v_n)$. The image set $\mathrm{Im}(T)$ is spanned by $G = \{w_{k+1}, \ldots, w_n\}$. The Dimension Theorem implies that the set G is linearly independent. We extend the set G to a basis $\gamma = \{w_1 \ldots w_k, w_{k+1}, \ldots, w_n\}$ of W. As a result, $[T]_\beta^\gamma$ is a diagonal matrix in which the last k entry in the diagonal are 1 and the other entries are 0:

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

Question 5. (15 point) Find all $x \in \mathbb{R}$ such that $\det\left(\begin{bmatrix} 0 & 1 & 1 & x \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ x & 1 & 1 & 0 \end{bmatrix}\right) = 0.$

Solution: We have

$$\begin{aligned}
\det\left(\begin{bmatrix} 0 & 1 & 1 & x \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ x & 1 & 1 & 0 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 0 & 1 & 1 & x \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1-x & -x \end{bmatrix}\right) \\
&= 0 + (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 1 & x \\ 1 & -1 & 0 \\ 1 & 1-x & -x \end{bmatrix}\right) + 0 + 0 \\
&= -\det\left(\begin{bmatrix} 1 & 1 & x \\ 0 & -2 & -x \\ 0 & -x & -2x \end{bmatrix}\right) \\
&= -\left((-1)^{1+1}\det\left(\begin{bmatrix} -2 & -x \\ -x & -2x \end{bmatrix}\right) + 0 + 0\right) \\
&= x^2 - 4x.
\end{aligned}$$

We conclude that the determinant is zero if and only if $x = 0$ or $x = 4$.