M E T U
Department of Mathematics

Group	Fundamentals of Mathematics Final	List No.

Code : *Math 111* Acad. Year : *2013* Semester : *Fall* Instructor : *G.Ercan, S.Finashin, M.Kuzucuoğlu, Ö.Küçüksakallı.* Date : *January 21, 2014* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES 80 TOTAL POINTS

1	2	3	4	5	6		

1. **(16pts)** Give an example of a pair of sets A and B such that

- A is countably infinite and $A - B$ is finite.

  Solution: Let $A = \mathbb{N}$ and $B = \mathbb{N} - \{1, 2\}$. Then $A - B = \{1, 2\}$ is a finite set.

- B and $A - B$ are both countably infinite.

  Solution: Let $A = \mathbb{Z}$ and $B = \mathbb{N}$. Then $A - B = \{0, -1, -2, -3, \ldots\}$ is countably infinite.

- A, B and $A - B$ are all uncountable.

  Solution: Let $A = \mathbb{R}$ and $B = \{x \in \mathbb{R} : x > 0\}$. Then $A - B = \{x \in \mathbb{R} : x \leq 0\}$ is uncountable.

- A and B are both uncountable but $A - B$ is countable.

  Solution: Let $A = \mathbb{R}$ and $B = \mathbb{R} - \{1, 2\}$. Then $A - B = \{1, 2\}$ is countable.

2. (12pts) Let A, B, C and D be sets. Suppose that $A \sim B$ and $C \sim D$. Prove that $A \times C \sim B \times D$.

Solution: Since $A \sim B$, there exists a bijective function $f : A \to B$ and since $C \sim D$, there exists a bijective function $g : C \to D$. Consider the function $h : A \times C \to B \times D$ defined by $h(a, c) = (f(a), g(c))$.

The function h is injective as $h(a_1, c_1) = h(a_2, c_2)$ implies $f(a_1) = f(a_2)$ and $g(c_1) = g(c_2)$. Since f and g are both injective, we have $a_1 = a_2$ and $c_1 = c_2$.

The function h is surjective because if we are given $(b, d) \in B \times D$, then there exists $a \in A$ such that $f(a) = b$ and there exists $c \in C$ such that $g(c) = d$ since f and g are both surjective. As a result $h(a, c) = (f(a), g(c)) = (b, d)$.

3. (12pts) Prove that the following formula holds for all $n \in \mathbb{N}$.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: It is easy to see that the formula is true for $n = 1$ since $1^3 = \frac{1^2 \cdot 2^2}{4}$. Now suppose that the formula is true for some n. We want to show that the formula is true for $n + 1$. Observe that

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 &= \tfrac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2 \left(\tfrac{n^2}{4} + (n+1) \right) \\ &= (n+1)^2 \left(\tfrac{n^2+4n+4}{4} \right) \\ &= \tfrac{(n+1)^2(n+2)^2}{4}. \end{aligned}$$

Hence the result is true for $n + 1$. Then by PMI, it is true for all $n \in \mathbb{N}$.

4. (12pts) Prove that $3^n > n^2 + n + 10$ for all natural numbers $n \geq 3$.

Solution: The inequality is true for $n = 3$ since $27 > 22 = 3^2 + 3 + 10$. Now suppose that the inequality is true for some natural number $n \geq 3$. We want to show that the inequality is true for $n + 1$. Observe that

$$\begin{aligned}
3^{n+1} &= 3 \cdot 3^n \\
&> 3(n^2 + n + 10) \\
&> n^2 + 3n + 12 \\
&= (n+1)^2 + (n+1) + 10.
\end{aligned}$$

Hence the result is true for $n + 1$. Then by PMI, it is true for all natural numbers $n \geq 3$.

5. (12pts) Let $f : A \to B$ and $X \subseteq A$. If f is bijective then prove that $f(A - X) = B - f(X)$.

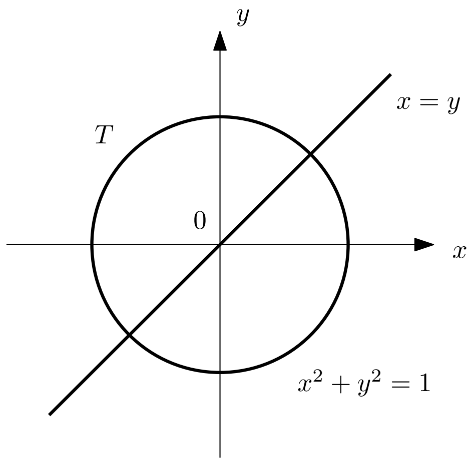
Solution: (Proof of $\subseteq$) Let b be an element of $f(A-X)$. Then there exists $a \in A-X$ such that $b = f(a)$. We want to show that b is not element of $f(X)$. Assume otherwise, then there exist $x \in X$ such that $b = f(x)$. Note that $a \notin X$ and $x \in X$ but their images are the same. This is a contradiction to the fact that f is injective.

(Proof of $\supseteq$) Let b be an element of $B - f(X)$. Since f is surjective, there exists $a \in A$ such that $b = f(a)$. Since $b = f(a)$ is not an element of $f(X)$, we conclude that $a \notin X$. Therefore $a \in A - X$ and as a result $b \in f(A - X)$.

6. **(16pts)** Let $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y \text{ or } x^2 + y^2 = 1\}$.

- Sketch the set T in the xy-plane.

  Solution:



- Is T a function from $\mathbb{R}$ to $\mathbb{R}$?

  Solution: No. Consider $x = 3/5$. Then $x\ T\ y$ is true for $y = -4/5, 3/5, 4/5$. Thus T is not a function of x. Similarly T is not a function of y.

- Is T a partial order on $\mathbb{R}$?

  Solution: No. The relation T is not anti-symmetric. Note that $1\ T\ 0$ and $0\ T\ 1$ but $0 \neq 1$.

- Is T an equivalence relation on $\mathbb{R}$?

  Solution: No. The relation T is not transitive. Note that $-1\ T\ 0$ and $0\ T\ 1$ but -1 and 1 are not related under T.