SUGGESTED EXERCISES FOR MIDTERM 2
MATH 111- FALL 2010 - METU

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The aim of this exercise set is to help you to study to the 2nd midterm of Math 111. The exam content is as follows:

- Cartesian Products and Relations
- Functions
- Equivalence Relations
- Equipollence and Countability

The main text for this course is your instructor's lecture notes. However you may want to consult the indicated sections of references for further information.

1. Cartesian Products and Relations

- Velleman 4.1, 4.2, 4.3
- Bloch 3.3, 5.1

Question 1.1. Let A and B be sets such that $B \subseteq A$. Prove that

$$A \times A \setminus B \times B = [(A \setminus B) \times A] \cup [A \times (A \setminus B)].$$

Question 1.2. Let A, B and C be subsets of a universal set U. Is it true that

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)?$$

Give either a proof or a counterexample to justify your answer.

Question 1.3. Let R be a relation from A to B and S be relation from B to C. Consider the sets

$$X = \{b \in B : \exists a \in A(aRb)\} \quad \text{and} \quad Y = \{b \in B : \exists c \in C(bSc)\}.$$

Prove that the composition of S and R, namely $S \circ R$, is empty if and only if X and Y are disjoint.

Question 1.4. Suppose that $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6, 7, 8\}$. Define the relations $R = \{(1, 7), (3, 6), (3, 7)\}$ and $S = \{(4, 7), (4, 8), (5, 6)\}$. Note that R is a relation from A to C and S is a relation from B to C. Find the following relations:

- $S^{-1} \circ R$.
- $R^{-1} \circ S$.

2. Functions

- Velleman 5.1, 5.2, 5.3, 5.4
- Bloch 4.1, 4.2, 4.3, 4.4

Question 2.1. Let A be a set and let $\mathcal{P}(A)$ be its power set. Define a function

$$f : \mathcal{P}(A) \to \mathcal{P}(A)$$

by the formula $f(X) = A \setminus X$. Is f injective, surjective?

Question 2.2. For each of the following functions, determine whether it is injective and determine its range:

- $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 2x + 1.$
- $f : \mathbb{Q} \to \mathbb{Q},\ f(x) = 2x + 1.$
- $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^x + 1.$
- $f : [0, \pi/2] \to \mathbb{R},\ f(x) = \sin(2x).$
- $f : [0, \pi/2] \to \mathbb{R},\ f(x) = \cos(2x).$

Question 2.3. Consider the relation S on real numbers $\mathbb{R}$ given by

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}.$$

- Is the logical sentece
$$\forall x \exists y (xSy)$$
true or false (the universe of discourse is $\mathbb{R}$)? Give either a proof or a counter-example to justify your answer.
- Is S a function from $\mathbb{R}$ to $\mathbb{R}$?

Question 2.4. Show that composition of two functions is a function.

Question 2.5. Let f be a function from A to B and let X be a subset of A. The *image* of X under f is given by $f(X) = \{f(x) : x \in X\}$. Consider the function from the power set $\mathcal{P}(A)$ to the power set $\mathcal{P}(B)$ induced by images

$$\begin{array}{rccc} F: & \mathcal{P}(A) & \to & \mathcal{P}(B) \\ & X & \mapsto & f(X). \end{array}$$

If X_1 and X_2 are arbitrary subsets of A, then show that

- $F(X_1 \cup X_2) = F(X_1) \cup F(X_2).$
- $F(X_1 \cap X_2) \subseteq F(X_1) \cap F(X_2).$ Give an example of a function f, where this inclusion is proper.

Question 2.6. Let f be a function from A to B and let Y be a subset of B. The *preimage* of Y under f is given by $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$. Consider the function from the power set $\mathcal{P}(B)$ to the power set $\mathcal{P}(A)$ induced by preimages

$$\begin{array}{rccc} G: & \mathcal{P}(B) & \to & \mathcal{P}(A) \\ & Y & \mapsto & f^{-1}(Y). \end{array}$$

If Y_1 and Y_2 are arbitrary subsets of B, then show that

- $G(Y_1 \cap Y_2) = G(Y_1) \cap G(Y_2).$
- $G(Y_1 \cup Y_2) = G(Y_1) \cup G(Y_2).$
- $G(Y_1^{\mathrm{c}}) = G(Y_1)^{\mathrm{c}}.$
- $G(Y_1 \setminus Y_2) = G(Y_1) \setminus G(Y_2).$

Question 2.7. Let f be a function from A to B. A function $g : B \to A$ is called a *right inverse* of f if $f \circ g = \mathrm{id}_B$, and it is called a *left inverse* of f if $g \circ f = \mathrm{id}_A$.

- Show that f has a right inverse if and only if it is surjective.
- Show that f has a left inverse if and only if it is injective.

Question 2.8. Suppose $f : A \to B$ and $g : B \to C$.

- Prove that if $g \circ f$ is onto then g is onto.
- Prove that if $g \circ f$ is one-to-one then f is one-to-one.

3. Equivalence Relations

- Velleman 4.6
- Bloch 5.3

Question 3.1. Let S be a relation on B. Define a relation R on A as follows:

$$R = \{(x, y) \in A \times A : (f(x), f(y)) \in S\}.$$

- Prove that if S is reflexive, then so is R.
- Prove that if S is symmetrice, then so is R.
- Prove that if S is transitive, then so is R.

Question 3.2. Let A be a set, and let R be a relation on A. Define the relation R' on A by $R' = (A \times A) \setminus R$.

- If R is reflexive, is R' necessarily reflexive, necessarily not reflexive or not necessarily either?
- If R is symmetric, is R' necessarily symmetric, necessarily not symmetric or not necessarily either?
- If R is transitive, is R' necessarily transitive, necessarily not transitive or not necessarily either?

Question 3.3. If x and y are integers we say that x divides y if there exists an integer k such that $y = kx$. Define

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \text{ divides } y\}.$$

Note that R is a relation on $\mathbb{Z}$ (Why?). Is R reflexive, symmetric, transitive?

Question 3.4. For each positive real number r, let $D_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < r\}$. Is D_r reflexive, symmetric, transitive?

Question 3.5. Suppose R is an equivalence relation on A, and let $g : A \to A/R$ be defined by the formula $g(x) = [x]_R$.

- Show that g is onto.
- Show that g is one-to-one if and only if $R = \mathrm{id}_A$.

Question 3.6. Let $f : A \to B$ be a function. Define a relation on A by letting xRy if and only if $f(x) = f(y)$, for all $x, y \in A$.

- Show that R is an equivalence relation.
- What can be said about the equivalence classes $[x]$ of R, depending upon whether f is injective but not surjective, surjective but not injective, neither or both?

Question 3.7. Suppose R and S are equivalence relations on A and $A/R = A/S$. Prove that $R = S$.

Question 3.8. Let T be a relation on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)T(z, w)$ if and only if

$$x^2 + y^2 = z^2 + w^2.$$

Prove that T is an equivalence relation and describe briefly the corresponding partition on $\mathbb{R} \times \mathbb{R}$, namely $(\mathbb{R} \times \mathbb{R})/T$.

4. Equipollence and Countability

- Velleman 7.1, 7.2, 7.3
- Bloch 6.1

Question 4.1. We use the following notation for interval of real numbers. If a, b are real numbers and $a < b$, then

$$\begin{aligned}(a,b) &= \{x \in \mathbb{R} : a < x < b\}\\ (a,b] &= \{x \in \mathbb{R} : a < x \leq b\}\\ [a,b) &= \{x \in \mathbb{R} : a \leq x < b\}\\ [a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}\end{aligned}$$

By writing a bijection, show that

- $(0,1) \approx (-1,11)$.
- $[0,1] \approx [1,11]$.
- $(0,1) \approx \mathbb{R}$. (Hint: Use $\arctan(x)$.)
- $(0,1) \approx [0,1)$. (This is tricky! Work on this if you have solved all other problems.)

Question 4.2. Let A, B, C and D be sets. Suppose that $A \approx B$ and $C \approx D$. Prove that

$$A \times C \approx B \times D.$$

Question 4.3. Let A and B be sets and $\mathcal{P}(A), \mathcal{P}(B)$ be their power sets respectively. If $A \approx B$, then prove that $\mathcal{P}(A) \approx \mathcal{P}(B)$.

Question 4.4. Prove that the set of all irrational numbers has the same cardinality with $\mathbb{R}$. In other words show that $\mathbb{R} \approx \mathbb{R} \setminus \mathbb{Q}$. Conclude that irrational numbers are uncountable.

Question 4.5. Let A be an uncountable set and let T be a non-empty set. Show that $A \times T$ is uncountable.

Question 4.6. Let A, B and C be sets.

- Show that $A \precapprox A$.
- Show that if $A \precapprox B$ and $B \precapprox C$, then $A \precapprox C$.

Question 4.7. Suppose $A \precapprox B$ and $C \precapprox D$.

- Prove that $\mathcal{P}(A) \precapprox \mathcal{P}(B)$.
- Prove that $A \times C \precapprox B \times D$.

Question 4.8. Show that $(0,1) \approx [0,1)$ using Schröder-Bernstein Theorem.