METU, Fall 2010, Math 111, Section 1.

Homework 6

1. Show that
$$\sum_{a=1}^{n} \frac{1}{\sqrt{a}} > \sqrt{n}$$
for all $n \in \mathbb{N}$ such that $n \geq 2$.

2. Define a sequence by setting $r_0 = 1$ and $r_{n+1} = 4r_n + 7$ for all $n \in \mathbb{N}$. Prove that
$$r_n = \frac{1}{3}(10 \cdot 4^n - 7)$$
for all $n \in \mathbb{N}$.

3. Define a sequence by setting $a_0 = a_1 = 1$ and
$$a_n = \frac{1}{3}\left(a_{n-1} + \frac{2}{a_{n-2}}\right)$$
for all $n \in \mathbb{N}$ such that $n \geq 2$. Prove that $1 \leq a_n \leq 3/2$ for all $n \in \mathbb{N}$.

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a function. For each part, give either a proof or a counterexample to justify your answer.
   - There must exist $k \in \mathbb{N}$ such that $f(n) \leq f(k)$ for all $n \in \mathbb{N}$.
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5. Prove that $\sqrt[3]{5}$ is irrational.