METU, Fall 2010, Math 111, Section 1.

Homework 4

Throughout this homework, let f be a function from A to B. In other words

$$f : A \to B$$

1. A function $g : B \to A$ is called a *right inverse* of f if $f \circ g = \mathrm{id}_B$. Show that the function f has a right inverse if and only if it is surjective.

2. Let X be a subset of A. Consider the function from the power set $\mathcal{P}(A)$ to the power set $\mathcal{P}(B)$ induced by images

$$\begin{array}{rccc} F: & \mathcal{P}(A) & \to & \mathcal{P}(B) \\ & X & \mapsto & f(X). \end{array}$$

   If X_1 and X_2 are arbitrary subsets of A, then show that

$$F(X_1 \cap X_2) \subseteq F(X_1) \cap F(X_2).$$

   Give an example of a function f, where this inclusion is proper.

3. Let Y be a subset of B. The *preimage* of Y under f is given by

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}.$$

   Consider the function from the power set $\mathcal{P}(B)$ to the power set $\mathcal{P}(A)$ induced by preimages

$$\begin{array}{rccc} G: & \mathcal{P}(B) & \to & \mathcal{P}(A) \\ & Y & \mapsto & f^{-1}(Y). \end{array}$$

   If Y_1 and Y_2 are arbitrary subsets of B, then show that

$$G(Y_1 \cap Y_2) = G(Y_1) \cap G(Y_2).$$

4. Let S be a relation on B. Define a relation R on A as follows:

$$R = \{(x, y) \in A \times A : (f(x), f(y)) \in S\}.$$

   - Prove that if S is reflexive, then so is R.
   - Prove that if S is symmetrice, then so is R.
   - Prove that if S is transitive, then so is R.

5. Suppose R and S are equivalence relations on A and $A/R = A/S$. Prove that $R = S$.