Make-up exam

Math 111 (Berkman, Küçüksakallı, Pamuk, Pierce)

January 22, 2011

Problem 1. Find either a disjunctive or a conjunctive normal form (DNF or CNF) of the propositional formula

$$(P \Rightarrow Q) \wedge R.$$

Problem 2. Are the following sets countable or uncountable? Explain briefly.

a. The set of points on a line in $\mathbb{R}^3$.

b. The set of points on a circle in $\mathbb{R}^2$.

c. The set of finite sequences of integers.

d. The set of algebraic numbers. (A real number is called algebraic if it is the root of a nonzero polynomial with integer coeficients.)

Problem 3. On $\mathbb{Z}$, define the relation E so that $x \mathrel{E} y$ if and only if the product xy is a *square* (that is, $xy = z^2$ for some z in $\mathbb{Z}$). It is known that, if $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

a. Show that E is an equivalence relation on $\mathbb{Z}$.

b. Determine, with justification, whether there are well-defined operations $\cdot$ and $+$ on $\mathbb{Z}/E$ given by

$$[x] \cdot [y] = [x \cdot y], \qquad [x] + [y] = [x + y].$$

Problem 4. Let $f\colon A \to B$. Prove or disprove:

a. If f is one-to-one, then the left inverse of f is unique.

b. If f is a bijection, then its inverse is unique.

Problem 5. Let $\leqslant_X$ and $\leqslant_Y$ be partial orderings on sets X and Y respectively. Define a new ordering $\leqslant$ on $X \times Y$ as follows:

$$(x_1, y_1) \leqslant (x_2, y_2) \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leqslant_Y y_2).$$

It is given that $\leqslant$ is a partial ordering: you need not prove this.

a. If $\leqslant_X$ and $\leqslant_Y$ are linear orderings, prove that $\leqslant$ also is a linear ordering.

b. If $\leqslant_X$ and $\leqslant_Y$ are well-orderings, prove that $\leqslant$ also is a well-ordering.

Problem 6. Define the integer sequence a_0, a_1, a_2, a_3, $\ldots$, recursively by

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 2, \qquad a_{n+3} = a_{n+2} + a_n.$$

Prove that $a_{n+2} \geqslant (\sqrt{2})^n$ for all n.

Problem 7. Prove or give a counterexample to each of the following statements with given universe of discourse.

- $\forall x\ \forall y\ ((x-2)(y^2+5) > 0)$, the universe of discourse is $\mathbb{R}$.
- $\forall x\ \exists y\ (3x + 4y = 5)$, the universe of discourse is $\mathbb{Q}$.
- $\exists x\ \exists y\ (x^2 - x = 2y + 1)$, the universe of discourse is $\mathbb{Z}$.