Name:

Student number:

1	
2	
3	
4	
5	
6	
7	
8	
Σ	

METU MATH 111, Final Exam

Monday, January 10, 2011, at 16:30 (120 minutes)

Instructors: Berkman, Küçüksakallı, Pamuk, Pierce

Instructions: There are 8 numbered problems on 4 pages. It should be obvious to the grader how to read your solutions. Please work carefully.

Problem 1. True or false? (Put T or F in the brackets. Do *not* justify your answers. Blank answers get 0; *wrong* answers get negative points.)

(a) $\mathbb{R}$ is countable .... (F)
(b) The set of polynomials in the variable x with integer coefficients is countable .. (T)
(c) If A and B are uncountable sets, then $A \smallsetminus B$ is countable .... (F)
(d) A relation R is symmetric if and only if $R = \breve{R}$ (that is, $R = R^{-1}$) .... (T)
(e) For all sets A and B, $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$ .... (T)
(f) For all sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ .... (F)
(g) $\neg(\exists x \in A)\ P(x)$ is logically equivalent to $(\forall x \in A)\ \neg P(x)$ .... (T)
(h) $((\forall x \in A)\ P(x)) \wedge ((\forall x \in B)\ P(x))$ is equivalent to $(\forall x \in (A \cup B))\ P(x)$ ...... (T)
(i) If A is a *totally* (or linearly) ordered set and x is a maximal element of A, then x is the maximum element of A .... (T)

Problem 2. The triangle inequality in $\mathbb{R}$ is given by $\forall x\ \forall y\ (|x+y| \leqslant |x| + |y|)$. Using this, prove that

$$|a_0 + a_1 + \cdots + a_n| \leqslant |a_0| + |a_1| + \cdots + |a_n|$$

for all nonempty finite sets $\{a_0, a_1, \ldots, a_n\}$ of real numbers.

Let $P(n)$ be the statement

$$|a_0 + \cdots + a_n| \leq |a_0| + \cdots + |a_n|$$

Since $|a_0| \leq |a_0|$ for any $a_0 \in \mathbb{R}$, $P(0)$ is true. Now suppose that $P(n)$ is true. Then

$$|\underbrace{a_0 + \cdots + a_n}_{x} + \underbrace{a_{n+1}}_{y}| \leq |a_0 + \cdots + a_n| + |a_{n+1}| \quad \text{(Triangle Inequality)}$$

$$\leq |a_0| + \cdots + |a_n| + |a_{n+1}| \quad \text{(Inductive Hypothesis)}$$

Thus $P(n+1)$ is true. By induction we conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 3. Suppose R is a partial ordering of A, and S is a partial ordering of B. Let

$$T = \{((a,b),(a',b')) \in (A \times B) \times (A \times B) \colon a\, R\, a' \text{ and } b\, S\, b'\}.$$

(a) Show that T is a partial ordering of $A \times B$.

T is reflexive: Since R and S are partial orders $a R a$ and bSb for a,b in A, B respectively. Thus $(a,b)T(a,b)$.

T is antisymmetric: If $(a,b)T(a',b')$ and $(a',b')T(a,b)$ then aRa' and $a'Ra$. Thus $a=a'$ since R is antisym. Similarly $b=b'$. Therefore T is antisymmetric.

T is transitive: If $(a,b)T(a',b')$ and $(a',b')T(a'',b'')$, then aRa' and $a'Ra''$. Thus aRa'' since R is transitive. Similarly bSb''. Therefore $(a,b)T(a'',b'')$ and T is transitive.

(b) If both R and S are *total* (or linear) orderings, will T also be a total ordering? Justify your answer.

No. Consider $A=B=\mathbb{R}$ and "less than or equal" relation on both A and B. The induced relation T is not a total order because neither $(1,2)T(2,1)$ nor $(2,1)T(1,2)$.

Problem 4. Suppose $b \in \mathbb{N}$, and A is a nonempty subset of $\mathbb{N}$ such that, for every element x of A, we have $x \leqslant b$. Prove that A has a maximum element with respect to $\leqslant$.

Consider $C=\{b-x : x\in A\}$, a subset of $\mathbb{N}$ since $x\leq b$ for all $x\in A$. Using well ordering principle pick $b-a_0 \in C$ a smallest element of C. Then $b-a_0 \leq b-x$ for all $x \in A$. It follows that $x\leq a_0$ for all $x\in A$. The element a_0 is a maximum element of A with respect to $\leq$.

Problem 5. Define the relation E on $\mathbb{R}$ by $x\ E\ y \iff x - y \in \mathbb{Z}$.

(a) Show that E is an equivalence relation.

E is reflexive: For any real x, we have $x - x = 0 \in \mathbb{Z}$. Thus $x E x$. The relation E is reflexive

E is symmetric: If xEy, then $x-y=k$ for some $k \in \mathbb{Z}$. Then $y-x=-k \in \mathbb{Z}$. So yEx and E is symmetric

E is transitive: If xEy and yEz, then we have $x-y=k$ and $y-z=\ell$ for some $k,\ell \in \mathbb{Z}$. Now $x-z$ equals $k+\ell$, an element of $\mathbb{Z}$. Therefore xEz and E is transitive

(b) What is [0]? $[0] = \{x \in \mathbb{R} : x - 0 \in \mathbb{Z}\} = \mathbb{Z}$

(c) Write three distinct elements of $\mathbb{R}/E$. $[0], [1/3], [2/3]$

(d) Is $\mathbb{R}/E$ countable? (Explain briefly.)

For any $x \in [0,1)$, we obtain a distinct equivalence class of E. Since $[0,1)$ is uncountable, $\mathbb{R}/E$ is uncountable too.

Problem 6. Let S be the relation $\{(x, y) \in \mathbb{R} \times \mathbb{R} \colon x^2 + y^2 = 1\}$ on $\mathbb{R}$. Answer each of the following questions by giving either a proof or a counterexample.

(a) Is the logical sentence $\forall x\ \exists y\ (x\ S\ y)$ true in $\mathbb{R}$?

No. If $x=2$, then there is no $y \in \mathbb{R}$ s.t $x^2+y^2=1$.

(b) Is S a function from $[-1, 1]$ to $\mathbb{R}$?

No. If $x=0$, then consider $y_1=1$ and $y_2=-1$. We have xSy_1 and xSy_2 but $y_1 \neq y_2$.

Problem 7. Explain briefly whether there are there propositional formulas $\boldsymbol{F}$ and $\boldsymbol{G}$ such that:

$$(P_0 \,\&\, P_1) \vee \boldsymbol{F} \sim P_0 \vee P_1, \qquad (P_0 \vee P_1) \vee \boldsymbol{G} \sim P_0 \,\&\, P_1$$

If F is the formula $P_0 \vee P_1$ then the former equivalence holds.

There is no propositional formula G satisfying the latter equivalence. To see this consider $P_0 = 0$ ad $P_1 = 1$. Then the left hand side is always 1 whereas the right hand side is always 0.

Problem 8. Do the the following equations (where a and b range over $\mathbb{Z}^+$) define functions f and g from $\mathbb{Q}^+$ to $\mathbb{Q}^+$? Justify your answers.

$$f\left(\frac{a}{b}\right) = a + b, \qquad g\left(\frac{a}{b}\right) = \frac{2a^2 + ab + b^2}{2b^2}.$$

The former equation does not define a function because $\frac{1}{2} = \frac{2}{4}$ in $\mathbb{Q}^+$ but

$$f\left(\tfrac{1}{2}\right) = 1+2 \neq 2+4 = f\left(\tfrac{2}{4}\right).$$

We can rewrite the latter equation as follows:

$$g\left(\tfrac{a}{b}\right) = \left(\tfrac{a}{b}\right)^2 + \tfrac{1}{2}\left(\tfrac{a}{b}\right) + \tfrac{1}{2}$$

Changing $\frac{a}{b}$ with $\frac{c}{d}$ (where $ad = bc$) has no effect on the value of the right hand side. Hence g defines a function from $\mathbb{Q}^+$ to $\mathbb{Q}^+$. Indeed we have $g(x) = x^2 + \frac{1}{2}x + \frac{1}{2}$ for all $x \in \mathbb{Q}^+$