METU
Department of Mathematics

	Fundamentals of Mathematics Midterm 2	
Code : *Math 111* Acad. Year : *2017 Fall* Instructor : *G.Ercan, S.Finashin* *M.Kuzucuoğlu, Ö.Küçüksakallı,* *F.Özbudak* Date : *December 19, 2017* Time : *17:40* Duration : *100 minutes*	Last Name : Name : Student No. : Department : Section : Signature :	
	5 QUESTIONS ON 4 PAGES 100 TOTAL POINTS	

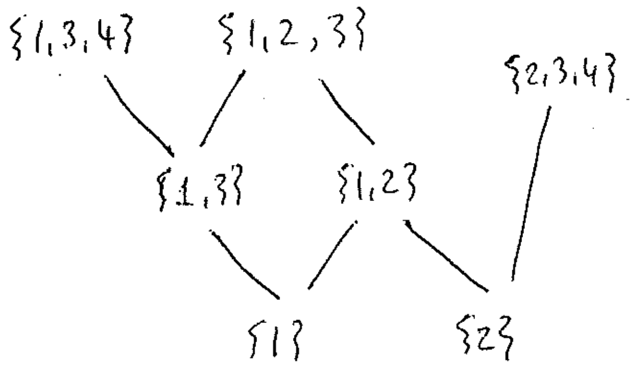
1	2	3	4	5		

READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

1. (20pts) Let $\mathcal{S} = \{\{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{2,3,4\}, \{1,3,4\}\}$. Let R be the partial order on $\mathcal{S}$ given by

$$A\,R\,B \iff A \subseteq B.$$

Draw the Hasse diagram of R. Determine whether each of the following types of elements of $\mathcal{S}$ exists and find all such elements in the case of existence: maximal, minimal, greatest, least. For $X = \{\{1,2\},\{1,3\}\}$ determine all greatest lower bounds and all least upper bounds of X if they exist.



Maximal : $\{1,3,4\}$, $\{1,2,3\}$ and $\{2,3,4\}$

Minimal : $\{1\}$ and $\{2\}$

Greatest : No such element

Least : No such element

Greatest lower bound for X : $\{1\}$

Least upper bound for X : $\{1,2,3\}$

2. (20pts) Let A, B be nonempty sets and $f : A \to B$ be a function. Prove the following statements:

(a) f is injective if and only if $E = f^{-1}(f(E))$ for all subsets $E \subseteq A$.

($\Rightarrow$) Suppose that f is 1-1.

Let $E \subseteq A$, and $a \in E$. Then $f(a) \in f(E)$ and so $a \in f^{-1}(f(E))$. Thus $E \subseteq f^{-1}(f(E))$

Let $a \in f^{-1}(f(E))$. Then $f(a) \in f(E)$ and hence $\exists x \in E$ s.t. $f(x) = f(a)$.

Since f is 1-1, we get $x = a \in E$. Thus $f^{-1}(f(E)) \subseteq E$

($\Leftarrow$) Suppose that $f^{-1}(f(E)) = E$ for all $E \subseteq A$.

Let $a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2)$.

Now $\{a_1\} = f^{-1}(f(\{a_1\})) = f^{-1}(f(\{a_2\})) = \{a_2\}$ and so $a_1 = a_2$.

It follows that f is 1-1

(b) f is surjective if and only if $F = f(f^{-1}(F))$ for all subsets $F \subseteq B$.

($\Rightarrow$) Suppose f is onto. Let $F \subseteq B$ and $b \in F$. Then $\exists a \in A$ s.t. $f(a) = b$ as f is onto. Now, $a \in f^{-1}(b) \subseteq f^{-1}(F)$ and so $b = f(a) \in f(f^{-1}(F)$

This shows that $F \subseteq f(f^{-1}(F))$

Let now $b \in f(f^{-1}(F))$. Then $\exists a \in f^{-1}(F)$ s.t. $f(a) = b \in F$

Therefore we have $f(f^{-1}(F)) \subseteq F$

($\Leftarrow$) Suppose that $F = f(f^{-1}(F))$ for all $F \subseteq B$. Let $b \in B$

Then $f(f^{-1}(\{b\})) = \{b\}$ and so $\exists a \in f^{-1}(\{b\})$.

Thus $f(a) = b$.

3. **(20pts)** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 - x$. Let R be the relation on $\mathbb{R}$ given by

$$x \, R \, y \iff (y = x \text{ or } y = f(x)).$$

Prove that R is an equivalence relation. Find the equivalence classes $[0]$ and $[1/3]$.

We need to show R is reflexive, symmetric and transitive

- R is reflexive: because $\forall x \in \mathbb{R}$ $x = x$ and hence xRx
- R is symmetric: Suppose xRy for $x, y \in \mathbb{R}$. Then $x = y$ or $y = 1 - x$.

It follows that $y = x$ or $x = 1 - y = f(y)$. Thus yRx holds.

- R is transitive: Suppose that xRy and yRz for $x, y, z \in \mathbb{R}$

Now $xRy \Rightarrow x = y$ or $y = 1 - x$ and $yRz \Rightarrow y = z$ or $z = 1 - y$

case 1: $x = y$ and $y = z$: Then $x = z$ and hence xRz

case 2: $x = y$ and $z = 1 - y$: Then $z = 1 - x$ and hence xRz

case 3: $y = 1 - x$ and $z = y$: Then $z = 1 - x$ and hence xRz

case 4: $y = 1 - x$ and $z = 1 - y$: then $z = 1 - y = 1 - (1 - x) = x$ and hence xRz

$[0] = \{x \in \mathbb{R} \mid xR0\} = \{x \in \mathbb{R} \mid x = 0 \text{ or } f(x) = 0\} = \{x \in \mathbb{R} \mid x = 0 \text{ or } x = 1\} = \{0, 1\}$

$[1/3] = \{x \in \mathbb{R} \mid xR\,1/3\} = \{x \in \mathbb{R} \mid x = 1/3 \text{ or } f(x) = 1/3\} = \{x \in \mathbb{R} \mid x = 1/3 \text{ or } x = 2/3\}$
$= \{1/3, 2/3\}$

4. **(20pts)** Let A be a set with at least two elements. Let $\mathcal{P}(A)$ denote the power set of A. Define the relation R on $\mathcal{P}(A)$ as follows: Let $X, Y \subseteq A$. We have

$$X \, R \, Y \iff X \cap Y = \emptyset.$$

Determine whether R is reflexive, symmetric, anti-symmetric and transitive.

not reflexive: as $X = \{1\}$ $X \not R X$ since $X \cap X = X \neq \emptyset$

symmetric: XRY implies $X \cap Y = \emptyset = Y \cap X$ so YRX

not antisymmetric: as $X = \{1\}$, $Y = \{2\}$ $X \cap Y = \emptyset$ and $Y \cap X = \emptyset$

i.e YRX and XRY but $X = \{1\} \neq \{2\} = Y$

~~transitive~~:

not transitive: $X = \{1\}$, $Y = \{2\}$, $Z = \{1\}$

XRY, since $X \cap Y = \emptyset$

YRZ, since $Y \cap Z = \emptyset$.

But $X \cap Z = \{1\} \neq \emptyset$ so $X \not R Z$.

5. (20pts) For each of the following functions determine if it is injective, surjective, bijective or neither.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \sqrt{x^2 + y^2}$.

not injective: $f(x,y) = f(x,-y) = f(-x,y) = f(-x,-y) = \sqrt{x^2+y^2}$

not surjective: $\sqrt{x^2+y^2} \geq 0 \Rightarrow f$ does not take negative values.

not bijective: because not injective. ~~as~~
(because not surjective)

(b) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x + y, x - y)$.

injective: if $g(x,y) = g(x_1,y_1)$, then $\left.\begin{array}{l} x+y = x_1+y_1 \\ x-y = x_1-y_1 \end{array}\right\}$ $\begin{array}{l} 2x = 2x_1 \\ 2y = 2y_1 \end{array}$

$\Rightarrow x = x_1$ and $y = y_1$

surjective: for any $(a,b) \in \mathbb{R}^2$, there exists $(x,y) \in \mathbb{R}^2$ s.t.
$g(x,y) = (a,b)$

$$\begin{array}{l} x+y = a \\ x-y = b \end{array} \Rightarrow \begin{array}{l} 2x = a+b \\ 2y = a-b \end{array} \Rightarrow \begin{array}{l} x = \frac{a+b}{2} \\ y = \frac{a-b}{2} \end{array}$$

Thus, g is bijective (injective and surjective)

(c) $h : \mathbb{R} \to \mathbb{R}^2$, $h(x) = (x + 1, x - 1)$.

injective: $h(x) = h(x_1) \Rightarrow \begin{array}{l} x+1 = x_1+1 \\ x-1 = x_1-1 \end{array} \Rightarrow x_1 = x$

not surjective: If $h(x) = (x+1, x-1) = (a,b)$, then $a-b=2$

so, if $a-b \neq 2$ (for example, for $(a,b) = (0,0)$, there is no x such that $h(x) = (a,b)$)

not bijective: because not surjective