M E T U
Department of Mathematics

	Fundamentals of Mathematics **Final Exam**	

Code : *Math 111* Acad. Year : *2017 Fall* Instructor : *G.Ercan, S.Finashin, M.Kuzucuoğlu, Ö.Küçüksakallı, F.Özbudak* Date : *January 16, 2018* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 4 PAGES 100 TOTAL POINTS

1	2	3	4	5	6	7	8		

1. (15pts) Consider the sequence s_n of natural numbers given by $s_1 = 5$ and $s_2 = 13$ and $s_n = 5s_{n-1} - 6s_{n-2}$ for all $n \geq 3$. Show that $s_n = 2^n + 3^n$ for all $n \geq 1$.

Solution: For $n = 1$ and $n = 2$, the formula $s_n = 2^n + 3^n$ is valid since we have $s_1 = 2^1 + 3^1 = 5$ and $s_2 = 2^2 + 3^2 = 13$, respectively. Suppose that the formula $s_n = 2^n + 3^n$ holds for all $n \in \{1, \ldots, k\}$ for some positive integer $k \geq 2$. We have

$$\begin{aligned} s_{k+1} &= 5s_k - 6s_{k-1} \\ &= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1}) \\ &= 2^k(5 - 3) + 3^k(5 - 2) \\ &= 2^{k+1} + 3^{k+1}. \end{aligned}$$

We see that the formula $s_n = 2^n + 3^n$ holds for $n = k + 1$. We conclude that the formula $s_n = 2^n + 3^n$ holds for all $n \geq 1$ by induction.

2. (10pts) Show that the set $S = (\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{N})$ is countable.

Solution: We start with noting

$$S = (\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{N}) = (\mathbb{Z} \cap \mathbb{R}) \times (\mathbb{R} \cap \mathbb{N}) = \mathbb{Z} \times \mathbb{N}.$$

Recall that $\mathbb{Z}$ and $\mathbb{N}$ are both countable. As a result, $\mathbb{Z} \times \mathbb{N}$ is countable too. Therefore the set S is countable

3. (12pts) For each of the following, determine if it is TRUE or FALSE. Explain briefly.

(i) There exists an injective function $f : \mathbb{R} \to \mathbb{Q}$.

Solution: **FALSE**. Recall that $\mathbb{Q} \sim \mathbb{N}$. If there were such a function then $\mathbb{R}$ would be countable which is not the case.

(ii) The set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m = n^3\}$ is of the same cardinality as $\mathbb{Q}$.

Solution: **TRUE**. The set A is countable by being a subset of a countable set $\mathbb{N} \times \mathbb{N}$. Note that A is infinite, so it is countably infinite. Therefore $A \sim \mathbb{Q}$.

(iii) Let B and C be nonempty sets. If B is countable and if there exists a surjective function $g : B \to C$, then C is countable.

Solution: **TRUE**. Since B is countable, there exists a surjective function $\tilde{g} : \mathbb{N} \to B$. Consider the composite function $g \circ \tilde{g} : \mathbb{N} \to C$. The function $g \circ \tilde{g}$ is surjective by being a composition of surjective functions. It follows that C is countable.

(iv) Let D be an infinite subset of $\mathbb{N} \times \mathbb{N}$ and let E be an uncountable set. Then the union $D \cup E$ is uncountable.

Solution: **TRUE**. Assume that $D \cup E$ is countable. Being a subset of a countable set, E must be countable. This is a contradiction. Therefore $D \cup E$ must be uncountable.

4. (13pts) Prove that $2^{2n} - 1$ is divisible by 3 for any positive integer n.

Solution: For $n = 1$, it is obvious that $2^{2 \cdot 1} - 1 = 3$ is divisible by 3. Suppose that $2^{2k} - 1$ is divisible by 3 for some integer $k \geq 1$. We have

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 4 \cdot 2^{2k} - 1 \\ &= (3 \cdot 2^{2k}) + (2^{2k} - 1). \end{aligned}$$

The first part in this final sum is divisible by 3. Moreover the second part is also divisible by 3 by the induction hypothesis. This proves that $2^{2(k+1)} - 1$ is divisible by 3. Therefore $2^{2n} - 1$ is divisible by 3 for any positive integer n by induction.

5. (15pts) Suppose that $a^2 + b^2 = c^2$ for some natural numbers a, b and c. Such a triple (a, b, c) is called a Pythagorean triple. Show that the product abc is divisible by 3.

Solution: Suppose that $a^2 + b^2 = c^2$ for some natural numbers a, b and c. Assume that none of a, b and c are divisible by 3. Then

$$(\ a \equiv 1 \pmod 3 \quad \text{or} \quad a \equiv 2 \pmod 3 \ ) \Longrightarrow a^2 \equiv 1 \pmod 3$$
$$(\ b \equiv 1 \pmod 3 \quad \text{or} \quad b \equiv 2 \pmod 3 \ ) \Longrightarrow b^2 \equiv 1 \pmod 3$$
$$(\ c \equiv 1 \pmod 3 \quad \text{or} \quad c \equiv 2 \pmod 3 \ ) \Longrightarrow c^2 \equiv 1 \pmod 3$$

The equation $a^2 + b^2 = c^2$ must be satisfied modulo 3 as well. But $1 + 1 \equiv 1 \pmod 3$ which is the same as $2 \equiv 1 \pmod 3$ is impossible. Hence our assumption "none of a, b and c are divisible by 3" must be wrong. It follows that one of a, b or c is divisible by 3. Thus abc is divisible 3.

6. (10pts) If m and n are integers then show that $m^2 - 4n \neq 2$.

Solution: Assume that m and n are integers such that $m^2 - 4n = 2$. We will derive a contradiction. We have $m^2 = 4n + 2$. Thus m must be even. There exists an integer k such that $m = 2k$. We obtain

$$2 = m^2 - 4n = (2k)^2 - 4n = 4k^2 - 4n = 4(k^2 - n)$$

It follows that $2 = 4(k^2 - n)$. This is a contradiction since 2 is not divisible by 4.

7. (12pts) Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that $g \circ f$ is bijective.
(i) Show that f is injective.

Solution: We are given that $g \circ f$ is bijective. Thus $g \circ f$ is injective. Suppose that $f(x) = f(y)$ for some $x, y \in A$. Then $g(f(x)) = g(f(y))$. It follows that $x = y$ since $g \circ f$ is injective. We conclude that f is injective.

(ii) Show that g is surjective.

Solution: We are given that $g \circ f$ is bijective. Thus $g \circ f$ is surjective. Let c be an element of C. There exists an element $a \in A$ such that $g(f(a)) = c$ since $g \circ f$ is surjective. Set $b = f(a)$. Observe that $g(b) = c$. In summary, for all $c \in C$ there exists $b \in B$ such that $c = g(b)$. We conclude that g is surjective.

8 (13pts) Let T be the relation defined on $\mathbb{R} \times \mathbb{R}$ as follows: $(x_1, y_1)\ T\ (x_2, y_2)$ if and only if $2x_1 - y_1 = 2x_2 - y_2$. Prove that T is an equivalence relation. Sketch the equivalence classes $[(1,2)]$ and $[(2,1)]$ in the Cartesian coordinate system.

Solution: Let (x_1, y_1) be an element of $\mathbb{R} \times \mathbb{R}$. Observe that $2x_1 - y_1 = 2x_1 - y_1$. Thus $(x_1, y_1)\ T\ (x_1, y_1)$. Therefore T is **reflexive**. Suppose that $(x_1, y_1)\ T\ (x_2, y_2)$ for some elements (x_1, y_1) and (x_2, y_2) of $\mathbb{R} \times \mathbb{R}$. We have $2x_1 - y_1 = 2x_2 - y_2$. It follows that $2x_2 - y_2 = 2x_1 - y_1$ and therefore $(x_2, y_2)\ T\ (x_1, y_1)$. We conclude that T is **symmetric**. Suppose that $(x_1, y_1)\ T\ (x_2, y_2)$ and $(x_2, y_2)\ T\ (x_3, y_3)$ for some elements $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) of $\mathbb{R} \times \mathbb{R}$. Then $2x_1 - y_1 = 2x_2 - y_2$ and $2x_2 - y_2 = 2x_3 - y_3$. It follows that $2x_1 - y_1 = 2x_3 - y_3$ and therefore $(x_1, y_1)\ T\ (x_3, y_3)$. Thus T is **transitive**.

Observe that

$$[(1,2)] = \{(x,y) \in \mathbb{R}^2 : (1,2)\ T\ (x,y)\} = \{(x,y) \in \mathbb{R}^2 : 0 = 2x - y\},$$
$$[(2,1)] = \{(x,y) \in \mathbb{R}^2 : (2,1)\ T\ (x,y)\} = \{(x,y) \in \mathbb{R}^2 : 3 = 2x - y\}.$$

These equivalence classes form parallel straight lines in the Cartesian coordinate system.