M E T U Department of Mathematics

MATH 371 Differential Geometry Spring-18 Midterm II 26.04.2018 17:40

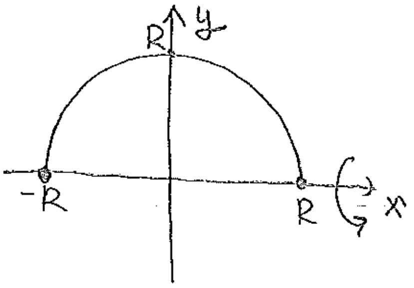
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	

4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1 25	2 45	3 10	4 20			

1. (12+13=25 pts) Let $S^2(R) = \{(x, y, z) \mid x^2 + y^2 + z^2 = R^2\}$ be the sphere with radius R centered at origin. Write parametrizations for $S^2(R)$ whose image covers $(0, 0, R)$ using the indicated methods.

a) Considering $S^2(R)$ as a surface of revolution of a semi-circle.



1st Method: $\alpha(u) = (R\cos(u), R\sin(u), 0)$ $0 \leq u < \pi$

$X(u,v) = (R\cos(u), R\sin(u)\cos v, R\sin(u)\sin v)$, $0 \leq v \leq 2\pi$

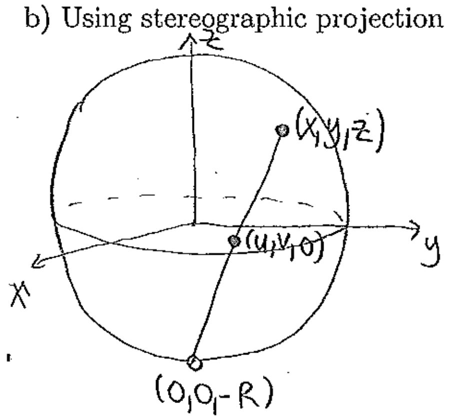
$X(\pi/2, \pi/2) = (0, 0, R)$

2nd Method: $\alpha(u) = (u, \sqrt{R^2-u^2}, 0)$ $-R \leq u \leq R$

$X(u,v) = (u, \sqrt{R^2-u^2}\cos v, \sqrt{R^2-u^2}\sin v)$, $0 \leq v \leq 2\pi$

$X(0, \pi/2) = (0, 0, R)$

b) Using stereographic projection from south pole $(0, 0, -R)$.



Consider the line

$l(t) = (0,0,-R) + t(u,v,R) = (tu, tv, tR-R)$

$R^2 = x^2+y^2+z^2 = (tu)^2 + (tv)^2 + (t-1)^2R^2$

$t^2u^2 + t^2v^2 + t^2R^2 - 2tR^2 + \cancel{R^2} = \cancel{R^2}$

$t^{\cancel{2}}(u^2+v^2+R^2) = 2\cancel{t}R^2$ ($t \neq 0$, gives the point on $S^2(R)$)

We get $t = \dfrac{2R^2}{u^2+v^2+R^2}$ which gives $x = \dfrac{2R^2 \cdot u}{u^2+v^2+R^2}$, $y = \dfrac{2R^2 v}{u^2+v^2+R^2}$

$z = \dfrac{2R^2}{u^2+v^2+R^2} \cdot R - R$

$X(u,v) = \left(\dfrac{2R^2u}{u^2+v^2+R^2}, \dfrac{2R^2v}{u^2+v^2+R^2}, \dfrac{R(R^2-u^2-v^2)}{u^2+v^2+R^2}\right)$

$X(0,0) = (0, 0, R)$

2. (7+13+12+13=45 pts) Let $D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2<1\}$. Consider the smooth map $\mathbf{x} : D \longrightarrow \mathbb{R}^3$ given by $\mathbf{x}(u,v) = (2u, 2v, 3\sqrt{1-u^2-v^2})$

a) Show that $M = \mathbf{x}(D)$ is a smooth surface.

1st Method: X is a coordinate patch, so M is a simple surface.

i) X is 1-1: $X(u_1,v_1) = (2u_1, 2v_1, 3\sqrt{1-u_1^2-v_1^2}) = (2u_2, 2v_2, 3\sqrt{1-u_2^2-v_2^2}) = X(u_2,v_2)$

$2u_1 = 2u_2 \Rightarrow u_1 = u_2$, $2v_1 = 2v_2 \Rightarrow v_1 = v_2$.

ii) X is regular: $X_u \times X_v = \left(2, 0, \frac{-3u}{\sqrt{1-u^2-v^2}}\right) \times \left(0, 2, \frac{-3v}{\sqrt{1-u^2-v^2}}\right)$

$= \left(\frac{+6u}{\sqrt{1-u^2-v^2}}, \frac{6v}{\sqrt{1-u^2-v^2}}, 4\right) \neq (0,0,0)$

2nd Method: Consider $g(x,y,z) = \frac{z^2}{9} + \frac{x^2}{4} + \frac{y^2}{4}$, $M \in g^{-1}(1)$

$dg = \frac{2x}{4}dx + \frac{2y}{4}dy + \frac{2z}{9}dz = 0 \Leftrightarrow x=y=z=0$. But, $(0,0,0) \notin g^{-1}(1)$

By Implicit Func. Thm, $M \subset g^{-1}(1)$ is a surface.

b) Let $p = (0,0,3) \in M$ and $\overrightarrow{w} = (2,-1,0) \in T_p\mathbb{R}^3$

(i) Find a basis for T_pM and show that $\overrightarrow{w}$ is a tangent vector at p.

$X(0,0) = (0,0,3)$. Hence, $\{X_u(0,0), X_v(0,0)\} = \{(2,0,0), (0,2,0)\}$ is a basis for T_pM.

$(2,-1,0)_p = 1 \cdot X_u(0,0) + \frac{1}{2} X_v$, so $\overrightarrow{w} \in T_pM$

(OR $(2,-1,0) \perp X_u(0,0) \times X_v(0,0)$)

(ii) Find a curve α lying on M such that $\alpha(0) = p$ and $\alpha'(0) = \overrightarrow{w}$

Take $\bar{\alpha}(t) = (0,0) + t(1, -\tfrac{1}{2})$, in D.

$\alpha(t) = X \circ \bar{\alpha}(t) = \left(2t, -t, 3\sqrt{1-t^2-\frac{t^2}{4}}\right)$

$\alpha(0) = (0,0,3) = p$

$\alpha'(0) = \left(2, -1, \frac{-\frac{15t}{4}}{\sqrt{1-\frac{5t^2}{4}}}\right)\Big|_{t=0} = (2,-1,0) = \overrightarrow{w}$

c) Determine the matrix of the shape operator S_p at $p = (0, 0, 3)$ of M.

$X_u = \left(2, 0, \frac{-3u}{\sqrt{1-u^2-v^2}}\right)$ $\quad X_u(0,0) = (2,0,0)$ $\quad E = 4, \; F = 0, \; G = 4$

$X_v = \left(0, 2, \frac{-3v}{\sqrt{1-u^2-v^2}}\right)$ $\quad X_v(0,0) = (0,2,0)$ $\quad U(0,0) = (0,0,1)$

$X_{uu} = \left(0, 0, \frac{-3\sqrt{1-u^2-v^2} + 3u \cdot \frac{(-2u)}{\sqrt{1-u^2-v^2}}}{1-u^2-v^2}\right)$ $\quad X_{uu}(0,0) = (0, 0, -3), \; L = -3$

$X_{uv} = (0, 0, 0)$ $\quad X_{uv}(0,0) = (0,0,0)$ $\quad M = 0$

$X_{vv} = \left(0, 0, \frac{-3\sqrt{1-u^2-v^2} + 3v \cdot \frac{(-2v)}{\sqrt{1-u^2-v^2}}}{1-u^2-v^2}\right)$ $\quad X_{vv}(0,0) = (0,0,-3)$ $\quad N = -3$

$$S_p = \mathrm{I}^{-1} \cdot \mathrm{II} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}$$

d) Compute the principal curvatures, the principal directions, the Gaussian curvature $K(p)$ and the mean curvature $H(p)$ at $p = (0, 0, 3)$ of M.

S_p is diagonal, so $-\frac{3}{4}$ is the only eigenvalue. Hence, $k_1 = k_2 = -\frac{3}{4}$ which implies that p is an umbilical point.

$\therefore \quad K = -\frac{3}{4} \cdot -\frac{3}{4} = \frac{9}{16}$

$H = \frac{-3/4 - 3/4}{2} = -3/4$

All directions are principal.

3. (10 pts) a) Let $M_c = \{(x, y, z) \mid x^3 - y^3 = c\}$. Determine all values of c so that M_c is a surface in $\mathbb{R}^3$.

Let $g(x,y,z) = x^3 - y^3$. Then, $dg = 3x^2dx - 3y^2dy + 0dz$

By Implicit Func. Thm, $g^{-1}(c) = M_c$ is a surface if $dg \neq 0$ on M_c.

$dg = 0 \Leftrightarrow \begin{matrix} 3x^2 = 0 \\ 3y^2 = 0 \end{matrix} \Leftrightarrow (x,y,z) = (0,0,z)$

$g(0,0,z) = 0$. Hence, $g^{-1}(c)$ is a surface for $c \neq 0$.

Let's check $c = 0$ case: $\quad x^3 - y^3 = 0 \Rightarrow x^3 = y^3 \Rightarrow x = y$

$x = y$ is a plane which is a surface in $\mathbb{R}^3$, too.

$\therefore$ M_c is a surface for all $c \in \mathbb{R}$.

4. (10+10=20 pts) This problem has two unrelated parts.

a) Find all smooth surface of revolutions in $\mathbb{R}^3$ with Gaussian curvature $K = 0$ everywhere. (Hint: You may assume that the surface is obtained by rotating $\alpha(u) = (f(u), g(u), 0)$ with unit-speed parametrization i.e. $f'(u)^2 + g'(u)^2 = 1$ around x-axis.

Assume the statement in the hint. Then, $X(u,v) = (f(u), g(u)\cos v, g(u)\sin v)$ $(g(u) > 0)$ is a parametrization of the surface of revolution. Now, let's compute Gaussian curvature K of it.

$X_u = (f'(u), g'(u)\cos v, g'(u)\sin v)$ $\quad X_u \times X_v = (g(u)\cdot g'(u), -f'(u)g(u)\cos v, -f'(u)g(u)\sin v)$

$X_v = (0, -g(u)\sin v, g(u)\cos v)$ $\quad \|X_u \times X_v\| = (g^2(u)g'^2(u) + f'^2(u)g^2(u))^{1/2} = g(u)$.

$X_{uu} = (f''(u), g''(u)\cos v, g''(u)\sin v)$ $\quad U = (g'(u), -f'(u)\cos v, -f'(u)\sin v)$

$X_{uv} = (0, -g'(u)\sin v, g'(u)\cos v)$

$X_{vv} = (0, -g(u)\cos v, -g(u)\sin v)$

$E = f'^2(u) + g'^2(u) = 1 \quad F = 0, \quad G = g^2(u)$

$L = g'(u)f''(u) - f'(u)g''(u), \ M = 0, \ N = f'(u)g(u)$

$$K = \frac{L\cdot N - M^2}{E\cdot G - F^2} = \frac{(g'(u)\cdot f'(u)f''(u) - f'^2(u)g''(u))g(u)}{g^2(u)}$$

$$\left(f'^2(u) + g'^2(u) = 1 \Rightarrow 2f'(u)f''(u) + 2g'(u)g''(u) = 0 \right)$$

$$= \frac{g'(-g'(u)\cdot g''(u)) - f'^2(u)g''(u)}{g(u)} = -\frac{g''(u)}{g(u)} = 0 \Rightarrow g''(u) = 0$$

Hence, $g(u) = au + b$, similarly $f(u) = cu + d$ with $a^2 + c^2 = 1$ which gives us a surface of revolution of line. This results in a cone or a cylinder.

b) Show that there is no point on a smooth surface with Gaussian curvature 20 and mean curvature 1.

Assume that there exists such a point p, $K(p) = 20 = k_1(p)\cdot k_2(p)$

$H(p) = 1 = \dfrac{k_1(p) + k_2(p)}{2}$

Then, $(k_1(p) + k_2(p))^2 = 4 = k_1^2(p) + k_2^2(p) + 2k_1(p)\cdot k_2(p) = k_1^2(p) + k_2^2(p) + 40$

Hence, $k_1^2(p) + k_2^2(p) = -36$ ↯