M E T U Department of Mathematics

MATH 371 Differential Geometry Midterm I 29.03.2018 17:40		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	

4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1 15	2 10	3 30	4 15	5 30			

1. (5+10=15 pts) Let $\alpha(s)$ be a unit speed curve with curvature $\kappa_\alpha(s)$ and positive torsion $\tau_\alpha(s)$. Let $\beta(s) = \int_0^s \mathbf{B}_\alpha(u)du$ where $\mathbf{B}_\alpha(s)$ is the binormal vector of α.

a) Show that $\beta(s)$ is a unit speed curve.

By Fundamental Theorem of Calculus, $\beta'(s) = B_\alpha(s)$ where $B_\alpha(s)$ is the unit binormal vector of curve α. Hence, $\|\beta'(s)\| = \|B_\alpha(s)\| = 1$

b) Show that β has curvature $\kappa_\beta(s) = \tau_\alpha(s)$, torsion $\tau_\beta(s) = \kappa_\alpha(s)$, unit tangent vector $\mathbf{T}_\beta(s) = \mathbf{B}_\alpha(s)$, unit normal vector $\mathbf{N}_\beta(s) = -\mathbf{N}_\alpha(s)$ and binormal vector $\mathbf{B}_\beta(s) = \mathbf{T}_\alpha(s)$.

① $T_\beta(s) = \beta'(s) = B_\alpha(s)$,

② $T_\beta'(s) = K_\beta(s) N_\beta(s) = \beta''(s) = B_\alpha'(s) = -\tau_\alpha(s) N_\alpha(s) \Rightarrow K_\beta(s) = \tau_\alpha(s), \ N_\beta(s) = -N_\alpha(s)$

③ $B_\beta(s) = T_\beta(s) \times N_\beta(s) = B_\alpha(s) \times (-N_\alpha(s)) = -(B_\alpha(s) \times N_\alpha(s)) = -(-T_\alpha(s)) = T_\alpha(s)$

④ $B_\beta'(s) = -\tau_\beta(s) N_\beta(s) = T_\alpha'(s) = K_\alpha(s) N_\alpha(s) \Rightarrow \tau_\beta(s) = K_\alpha(s)$

2. (10 pts) Let $\alpha(s)$ be a unit speed curve lying on a sphere of radius R centred at origin i.e. $||\alpha(s)||^2 = R^2$. Show that the curvature $\kappa(s)$ of the curve $\alpha(s)$ satisfies $\kappa(s) \geq \frac{1}{R}$.

We have $\alpha(s) \cdot \alpha(s) = R^2$. Then, $(\alpha(s) \cdot \alpha(s))' = 0$ where $(\alpha(s) \cdot \alpha(s))' = 2\alpha'(s) \cdot \alpha(s)$. Hence, $\alpha'(s) \cdot \alpha(s) = 0$. By taking one more derivative we get $\alpha'' \cdot \alpha(s) + \alpha'(s) \cdot \alpha'(s) = 0$. But $\alpha'(s) \cdot \alpha'(s) = 1$ since α is unit speed. So, $\alpha''(s) \cdot \alpha(s) = -1$. By Cauchy-Schwartz

$1 = |-1| = |\alpha''(s) \cdot \alpha(s)| \leq \|\alpha''(s)\| \|\alpha(s)\| = K(s) \cdot R$.

$\therefore \quad \frac{1}{R} \leq K(s)$

3. (5+10+8+7=30 pts) Consider the curve $\alpha(t) = (\sin(t^2), t^2+\sqrt{3}, \cos(t^2)+1)$, $t \in [0,\infty)$.

a) Write a unit speed parametrization $\beta(s)$ of this curve.

$\alpha'(t) = (\cos(t^2)\cdot 2t,\ 2t,\ -\sin(t^2)2t)$ $\quad s = \int_0^t \|\alpha'(u)\|\,du = \int_0^t 2\sqrt{2}\,u\,du = \sqrt{2}t^2$

$t = \sqrt{\frac{s}{\sqrt{2}}}$

$\beta(s) = \left(\sin(s/\sqrt{2}),\ \frac{s}{\sqrt{2}}+\sqrt{3},\ \cos\left(\frac{s}{\sqrt{2}}\right)+1\right)$

b) Compute the curvature and torsion of $\beta(s)$.

$\beta'(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right)\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\sin\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}}\right) = T(s)$

$\beta''(s) = \left(-\sin\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{2},\ 0,\ -\cos\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{2}\right)$

$K(s) = \|\beta''(s)\| = \sqrt{\sin^2\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{4} + \cos^2\left(\frac{s}{\sqrt{2}}\right)\frac{1}{4}} = \frac{1}{2}.$ $\quad \boxed{K(s) = \frac{1}{2}}$

$\beta''(s) = K(s)N(s) = \frac{1}{2}\left(-\sin\left(\frac{s}{\sqrt{2}}\right),\ 0,\ -\cos\left(\frac{s}{\sqrt{2}}\right)\right)$

$B(s) = T(s) \times N(s) = \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), & \frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \\ -\sin\left(\frac{s}{\sqrt{2}}\right), & 0, & -\cos\left(\frac{s}{\sqrt{2}}\right) \end{vmatrix} = \left(\frac{-1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\right)$

$B'(s) = -\tau(s)N(s) = \left(+\frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right),\ 0,\ \frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right)\right)$, $\boxed{\tau(s) = \frac{1}{2}}$

c) Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the map defined as $F(x,y,z) = ((x+z)/\sqrt{2}, +y, (x-z)/\sqrt{2})$. Define $\gamma(s) = F \circ \beta(s)$. Compute the curvature and torsion of $\gamma(s)$.

F is a linear map whose matrix with respect to standard basis is equal to $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & +\frac{1}{\sqrt{2}} \\ 0 & +1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$. It's an orthogonal matrix. Hence F is an isometry with determinant -1.

So, $K_\gamma(s) = K_\beta(s) = \frac{1}{2}$, $\quad \tau_\gamma(s) = \text{sign}\,F \cdot \tau_\beta(s) = -1\cdot\frac{1}{2} = -\frac{1}{2}.$

d) Find an isometry of $\mathbb{R}^3$, if it exists, which maps the curve $\beta(s)$ to a curve on the xy-plane. Otherwise, explain its non-existence.

If there were such an isometry, then the image curve would be a plane curve whose torsion would be equal to zero. But $\beta(s)$ (or $\alpha(t)$) has non-zero torsion which is preserved under isometries up to a sign. Hence, such an isometry can't exist.

4. (10+5=15 pts) a) Find the plane curve α parametrised by arclength with curvature $\kappa(s) = -2$ and $\alpha(0) = (1, 1)$ $\alpha'(0) = (-1, 0)$.

$$\cos\theta_0 = -1, \quad \sin\theta_0 = 0 \Rightarrow \theta_0 = \pi$$

$$\theta(s) = \theta_0 + \int_0^s \kappa(t)\,dt = \pi + \int_0^s -2\,dt = \pi - 2s$$

$$x(s) = x_0 + \int_0^s \cos(\theta(t))\,dt = 1 + \int_0^s \cos(\pi - 2t)\,dt = 1 + \left(\frac{-\sin(\pi-2t)}{2}\Big|_0^s\right)$$

$$= 1 - \frac{\sin(\pi - 2s)}{2}$$

$$y(s) = y_0 + \int_0^s \sin(\theta(t))\,dt = 1 + \int_0^s \sin(\pi - 2t)\,dt = 1 + \left(\frac{\cos(\pi-2t)}{2}\Big|_0^s\right)$$

$$= 1 + \left(\frac{\cos(\pi - 2s)}{2} + \frac{1}{2}\right) = \frac{3}{2} + \frac{\cos(\pi-2s)}{2}$$

$\alpha(s) = (x(s), y(s))$ which satisfies $(x-1)^2 + (y - 3/2)^2 = \left(\frac{1}{2}\right)^2$ i.e a circle.

b) Find the space curve β parametrised by arclength with curvature $\kappa(s) = 2$, torsion $\tau(s) = 0$ and $\beta(0) = (1, 1, 0)$ $\beta'(0) = (-1, 0, 0)$, $\beta''(0) = (0, 2, 0) = 2\cdot(0,1,0)$

β must be a plane curve since $\tau(s) = 0$. This plane has normal B and $B \parallel \beta' \times \beta'' \parallel (0,0,1)$. Hence, this plane must be parallel to xy-plane. Since $\beta(0)$ is on this plane, actually it must be the xy-plane. As a result, we just need to consider the first 2-coordinates of these initial conditions. Then, the solution found in (a) clearly satisfies all the given conditions.

5. (25+5=30 pts) This problem has unrelated parts.

a) Let $\phi = yx^2dx + xz^2dz$, $\quad \psi = 2yzdx + 3xzdy + 4xydz$

$V = -yU_1 + xU_3 \quad W = \cos xU_1 + \sin xU_2$ and $g(x, y, z) = x^2y + z\cos(y)$.

(i) Compute $d(\psi \wedge \phi)$.

$d(\psi \wedge \phi) = d\psi \wedge \phi - \psi \wedge d\phi$

$d\psi = (3z - 2z)dx \wedge dy + (4y - 2y)dx \wedge dz + (4x - 3x)dy \wedge dz$

$= z\,dx \wedge dy + 2y\,dx \wedge dz + x\,dy \wedge dz$

$d\phi = -x^2 dx \wedge dy + z^2 dx \wedge dz$

$d(\psi \wedge \phi) = (x z^3 + x^3 y)dx \wedge dy \wedge dz - (-4x^3y - 3xz^3)dx \wedge dy \wedge dz = (5x^3y + 4xz^3)dx \wedge dy \wedge dz$

(ii) Show that there is no smooth function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $df = \phi$.

If there were, $d(df) = 0$, but $d(df) = d\phi \neq 0$ by (a).

(iii) $V[g]$.

$(-yU_1 + xU_3)[x^2y + z\cos y] = -y(2xy) + x\cos y$

(iv) $W[\phi(V)](\pi, 1, 2)$

$\phi(V) = -y^2x^2 + x^2z^2$, $\quad \cos x\, U_1 + \sin x\, U_2 [-y^2x^2 + x^2z^2] = \cos x \cdot (-2xy^2 + 2xz^2) + \sin x(-2yx^2)$

$(-2xy^2 + 2xz^2)\cos x - 2x^2y\sin x \Big|_{(\pi,1,2)} = -6\pi.$

(v) $\nabla_V(\nabla_V W)$.

$\nabla_V W = -y(-\sin x U_1 + \cos x U_2) = +\sin x \cdot y U_1 + y\cos x U_2$

$\nabla_V(\nabla_V W) = -y(\cos x \cdot y U_1 - y\sin x U_2) = -y^2\cos x U_1 - y^2 \sin x U_2$

b) Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as $F(u, v) = (ve^u, 2u)$. Show that F is a diffeomorphism.

One to One: $(v_1e^{u_1}, 2u_1) = (v_2e^{u_2}, 2u_2) \Rightarrow u_1 = u_2,\ v_1 = v_2$

Onto: For any $(x,y) \in \mathbb{R}^2$, $F(y/2, xe^{-y/2}) = (xe^{-y/2} \cdot e^{y/2}, 2\, y/2) = (x,y)$

Regular: $J(F) = \begin{bmatrix} ve^u & e^u \\ 2 & 0 \end{bmatrix}$ has determinant $-2 \cdot e^u \neq 0$.

$\therefore$ F is a diffeomorphism.