M E T U Department of Mathematics

MATH 371 Differential Geometry Final Exam 01.06.2018 09:30		
Last Name : Name : Student No:	Signature : Duration : 150 *minutes*	
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

1 25	2 25	3 25	4 15	5 10			

1. (20 pts) Consider the curve $\alpha : (-\infty, +\infty) \longrightarrow \mathbb{R}^3$, given by

$$\alpha(t) = (t - \sqrt{3}\sin t,\ 2\cos t, \sqrt{3}t + \sin t)$$

Compute T, B and the curvature $\kappa(t)$, the torsion $\tau(t)$ of α.

$\alpha'(t) = (1 - \sqrt{3}\cos t, -2\sin t, \sqrt{3} + \cos t)$, $\alpha''(t) = (\sqrt{3}\sin t, -2\cos t, -\sin t)$

$\alpha'''(t) = (\sqrt{3}\cos t, 2\sin t, -\cos t)$

$$T(t) = \frac{(1-\sqrt{3}\cos t, -2\sin t, \sqrt{3}+\cos t)}{2\sqrt{2}}$$

$$B(t) = \frac{(2+2\sqrt{3}\cos t, +4\sin t, 2\sqrt{3}-2\cos t)}{4\sqrt{2}}$$

$$K(t) = \frac{4\sqrt{2}}{(2\sqrt{2})^3} = \frac{1}{4}$$

$$\tau(t) = \frac{8}{(4\sqrt{2})^2} = \frac{1}{4}.$$

2. (2+3=5 pts) a) State the definition of a geodesic.

A curve α in $M \subset \mathbb{R}^3$ is a geodesic of a surface M if its acceleration α'' is always normal to M, i.e. $[\alpha'']^T = 0$.

b) State Theorema Egregium

The Gaussion curvature is determined by only the first fundamental form. (OR, Under a local isometry Gaussian curvature is preserved.

3. **(4+4+7+7+8+10=40 pts)** Let $M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=z^2,\ z>0\}$.

a) Show that M is a surface by using Implicit Function Theorem.

Let $g(x,y,z) = x^2+y^2-z^2$, then $dg = 2x\,dx + 2y\,dy - 2z\,dz$

$M \subset g^{-1}(0)$. By Implicit Function Thm., M is a smooth surface if $dg|_M \neq 0$. $dg = 0 \Leftrightarrow (x,y,z)=(0,0,0) \notin M$.

$\therefore M$ is a smooth surface of $\mathbb{R}^3$.

b) Show that $\mathbf{x}(u,v) = (u\cos v, u\sin v, u)$ is a parametrization of M for $0<u$, $0 \le v \le 2\pi$.

1) $x = u\cos v$, $y = u\sin v$, $z=u$. We have $(u\cos v)^2 + (u\sin v)^2 = u^2$ ✓

2) $X_u = (\cos v, \sin v, 1)$, $X_v = (-u\sin v, u\cos v, 0)$

$\|X_u \times X_v\| = \sqrt{2\cdot u^2 - 0} = \sqrt{2u^2} = \sqrt{2}\,|u| > 0$ $\therefore X$ is regular.

Hence, $X(u,v)$ is a parametrization of M.

c) Compute the first fundamental form and show that M is flat. (<u>DO NOT</u> use the shape operator.)

$E = 2$, $F = 0$, $G = u^2$ $\quad I = \begin{bmatrix} 2 & 0 \\ 0 & u^2 \end{bmatrix}$

We can use 3^{rd} formula to compute K.

$$K = \frac{1}{2\sqrt{2\cdot u^2}}\left(\left(\frac{0}{\sqrt{2\cdot u^2}}\right)_v + \left(\frac{2u}{\sqrt{2u^2}}\right)_u\right) \underset{u>0}{=} \frac{1}{2\sqrt{2}u}\left(0 + \left(\frac{2\cancel{u}}{\sqrt{2}\cancel{u}}\right)_u\right) = 0$$

d) Compute the shape operator and verify that M is flat.

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{(-u\cos v, -u\sin v, u)}{\sqrt{2}u} = \frac{1}{\sqrt{2}}(-\cos v, -\sin v, 1)$$

$X_{uu} = (0,0,0)$, $X_{uv} = (-\sin v, \cos v, 0)$, $X_{vv} = (-u\cos v, -u\sin v, 0)$

$$II = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{u}{\sqrt{2}} \end{bmatrix} \qquad S = I^{-1}\cdot II = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{u^2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \frac{u}{\sqrt{2}} \end{bmatrix}$$

We get $S = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}u} \end{bmatrix}$ $\qquad K = \det(S) = 0$.

e) Compute the principal directions and principal curvatures of M at the point (1,0,1).

$X(u,v)=(1,0,1) \Rightarrow (u,v)=(1,0)$ or $(1,2\pi)$

$S_{(1,0,1)}=\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ Eigenvalues are $\{0, \frac{1}{\sqrt{2}}\}$ which are also principal curvatures

Eigenvectors are $\left\{k\begin{bmatrix}1\\0\end{bmatrix}\right\}$ for $\lambda=0$, $\left\{k\begin{bmatrix}0\\1\end{bmatrix}\right\}$ for $\lambda=\frac{1}{\sqrt{2}}$

So, principal directions are $X_u(1,0)=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $X_v(1,0)=\begin{bmatrix}0\\1\\0\end{bmatrix}$.

f) Write the geodesic equations by using the parametrization $\mathbf{x}(u,v)$ and show that none of the parallels is a geodesic.

$$\begin{bmatrix}\Gamma^u_{uu}\\ \Gamma^v_{uu}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{u^2}\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}\Gamma^u_{uv}\\ \Gamma^v_{uv}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{u^2}\end{bmatrix}\cdot\begin{bmatrix}0\\u\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{u}\end{bmatrix}$$

$$\begin{bmatrix}\Gamma^u_{vv}\\ \Gamma^v_{vv}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{u^2}\end{bmatrix}\cdot\begin{bmatrix}-u\\0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}u\\0\end{bmatrix}$$

. We get

1) $u''-\frac{1}{2}u\cdot(v')^2=0$

2) $v''+2\cdot\frac{1}{u}u'\cdot v'=0$

When $u=c$, we get parallels. If we plug into the equations,

$0-\frac{1}{2}c\cdot(v')^2\overset{?}{=}0$

$v''+2\cdot\frac{1}{c}\cdot 0\cdot v'=0 \Rightarrow v(t)=at+b$, $v'(t)=a\neq 0$ but 1^{st} equation is not satisfied.

(Note $a=0$ gives just a point!)

4. (5 + 5 = 10 pts) For each statement below, prove the statement if it is True, otherwise disprove by giving a counterexample.

a) Let $x(u,v)$ and $y(u,v)$ be two parametrizations for surfaces M_1 and M_2 on a domanid D, respectively. If $I_{x(u,v)}=I_{y(u,v)}$ for all (u,v), then M_1 and M_2 have the same Gaussian curvature.

TRUE If the first fundamental forms are same, by Theorema Egregium they will have the same Gaussian curvature at $X(u,v)$, and at $Y(u,v)$, respectively.

b) Show that if M is a flat minimal surface, then it must be a part of a plane.

TRUE If M is a flat minimal surface, then $k_1(p)\cdot k_2(p)=0$, $k_1(p)+k_2(p)=0$ for all $p\in M$.

Hence, $k_1(p)=k_2(p)=0$ i.e. M is flat and umbilic. which implies M is a plane.

5. **(15+10=25 pts)** This problem has two unrelated parts about geodesics.

a) Let $\mathbf{x}(u,v)$ be a coordinate patch for a smooth surface M with $F=0$. Prove that if all u-curves and v-curves are geodesics, then M is flat.

Let $I = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$ be the first fundamental form of X for X(u,v)

Then $\begin{bmatrix} \Gamma_{uu}^{u} \\ \Gamma_{uu}^{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \frac{1}{2}E_u \\ -\frac{1}{2}E_v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{E_u}{E} \\ -\frac{1}{2}\frac{E_v}{G} \end{bmatrix}.$

Similarly, $\begin{bmatrix} \Gamma_{uv}^{u} \\ \Gamma_{uv}^{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{E_v}{E} \\ \frac{1}{2}\frac{G_u}{G} \end{bmatrix}$, $\begin{bmatrix} \Gamma_{vv}^{u} \\ \Gamma_{vv}^{v} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{G_u}{E} \\ \frac{1}{2}\frac{G_v}{G} \end{bmatrix}$

Geodesic eqns: $u'' + \frac{1}{2}\frac{E_u}{E}(u')^2 + \frac{E_v}{E}u' \cdot v' - \frac{1}{2}\frac{G_u}{E}(v')^2 = 0$

$v'' - \frac{1}{2}\frac{E_v}{G}(u')^2 + \frac{G_u}{G}u' \cdot v' + \frac{1}{2}\frac{G_v}{G}(v')^2 = 0$

If u-curves ($v=c$) and v-curves ($u=c$) are geodesics, then we get $-\frac{1}{2}\frac{E_v}{G} = 0$ & $-\frac{1}{2}\frac{G_u}{E} = 0$ which implies $E_v = G_u = 0$.

By using 3^{rd} formula for K, we get $K=0$ everywhere.

b) Let M be a surface of revolution. Show that if all of its parallels are geodesics, then it must a circular cylinder.

Let M be a surface of revolution. Without loss of generality, we may assume M is obtained from $\alpha(u) = (f(u), g(u))$ on xz-plane by rotating around z-axis. Hence, $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ is a coordinate patch. As we know from lectures, a parallel $u=c$ is a geodesic if $f'(u) = f'(c) = 0$. If all parallels are geodesics, then $f'(u) = 0$ for all u. So, $f(u) = k \neq 0$ (constant)

Then, $X(u,v) = (k\cos v, k\sin v, g(u))$ which is a patch for a circular cylinder $x^2+y^2=k^2$ in $\mathbb{R}^3$.