## COMMUNICATIONS

# Critical Reynolds Number for Newtonian Flow in Rectangular Ducts

The critical Reynolds number is calculated as a function of the duct aspect ratio for flow of Newtonian fluids in ducts of rectangular cross sections. A simplified procedure is also presented using an approximate velocity distribution.

The problem of transition from laminar to turbulent flow is rather complex. Among the several theoretical methods available in the literature to determine the critical Reynolds number (Lin, 1955; Hanks, 1963, 1969; Spriggs, 1973), the method developed by Hanks seems to be the most promising.

Hanks (1963) argued that when the magnitude of the angular acceleration term reaches some scalar multiple, $K$, of the diffusional momentum flux vector the fluid motion will be unstable to certain types of disturbances and stable laminar flow will no longer exist. "A unique numerical value of the stability parameter, $K$, permits one to calculate the critical value of an arbitrarily defined Reynolds number regardless of such complicating factors as noncircular geometries, nonisothermal flows, or non-Newtonian fluid behavior", as stated by Hanks.

Hanks and Ruo (1966) calculated the lower critical Reynolds number for flow of Newtonian fluids in ducts of rectangular cross sections. However, the results of this analysis, as will be explained later, are in error. In this work, correct values of the critical Reynolds number are evaluated as a function of the duct aspect ratio. The comparison of the values with the ones evaluated by Hanks and Ruo indicates a considerable discrepancy.

The calculation of the critical Reynolds number requires the use of the fully developed laminar flow velocity distribution. Since this expression is complex for flow in a rectangular duct, computations become lengthy. A simplified procedure is also presented using an approximate velocity distribution.

### Theoretical Development

For steady-state, one-dimensional flow of an incompressible Newtonian fluid, the semiempirical stability parameter, $K$, is given by Hanks (1963) as

$$K = \rho\langle v_z\rangle^2 u \frac{([\partial u/\partial x]^2 + [\partial u/\partial y]^2)^{1/2}}{[-\mathrm{d}P/\mathrm{d}z]} \tag{1}$$

where

$$u = v_z/\langle v_z\rangle \tag{2}$$

Here $\rho$ is the density, $v_z$ is the $z$ component of the velocity, $\langle v_z\rangle$ is the average velocity, and $P$ is the pressure.

Before going into the application of eq 1 for flow in a rectangular duct, the introduction of the concept of hydraulic equivalent and equivalent diameters will be in order. Unfortunately, these two diameters have not been clearly differentiated in the literature. For flow in noncircular ducts, two types of Reynolds numbers can be defined with respect to these diameters.

The hydraulic equivalent diameter, $D_\mathrm{h}$, is defined by

$$D_\mathrm{h} = 4\frac{\text{flow area}}{\text{wetted perimeter}} \tag{3}$$

The Reynolds number, based on the hydraulic equivalent diameter, is

$$Re_\mathrm{h} = \frac{D_\mathrm{h}\langle v_z\rangle\rho}{\mu} \tag{4}$$

Therefore, the friction factor, based on the hydraulic equivalent diameter, can be related to $Re_\mathrm{h}$ in the form

$$f_\mathrm{h} = \Omega\frac{16}{Re_\mathrm{h}} \tag{5}$$

where $\Omega$ depends on the geometry of the system. Since $\Omega = 1$ only for a circular pipe, the use of the concept of hydraulic equivalent diameter has not been recommended for laminar flow (Bird et al., 1960; Fahien, 1983).

Lohrenz and Kurata (1960) proposed to use an equivalent diameter, $D_\mathrm{eq}$, given in the form

$$D_\mathrm{eq} = \left[\frac{32\mu\langle v_z\rangle}{(-\mathrm{d}P/\mathrm{d}z)}\right]^{1/2} \tag{6}$$

so that the relationship between friction factor and Reynolds number in any flow geometry can be represented by the same equation, i.e.,

$$f_\mathrm{eq} = 16/Re_\mathrm{eq} \tag{7}$$

where the Reynolds number, based on the equivalent diameter, is

$$Re_\mathrm{eq} = \frac{D_\mathrm{eq}\langle v_z\rangle\rho}{\mu} \tag{8}$$

Noting that

$$Re_\mathrm{eq}/Re_\mathrm{h} = f_\mathrm{eq}/f_\mathrm{h} = D_\mathrm{eq}/D_\mathrm{h} \tag{9}$$

the function $\Omega$ can be obtained by using eq 5 and 7 as

$$\Omega = [D_\mathrm{h}/D_\mathrm{eq}]^2 \tag{10}$$

By use of the preceding definitions, the stability parameter given in eq 1 can be rearranged in the form

$$K = \frac{D_\mathrm{eq}}{32}Re_\mathrm{eq}\phi(x,y) = \frac{D_\mathrm{h}}{32\Omega}Re_\mathrm{h}\phi(x,y) \tag{11}$$

where

$$\phi(x,y) = u([\partial u/\partial x]^2 + [\partial u/\partial y]^2)^{1/2} \tag{12}$$

The values of $\langle v_z\rangle$, $u$, $D_\mathrm{h}$, $D_\mathrm{eq}$, and $\Omega$ for flow between

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Table I. Values of $\langle v_z \rangle$, $u$, $D_h$, $D_{eq}$, and $\Omega$ for Flow between Parallel Plates and Rectangular Ducts

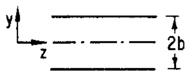
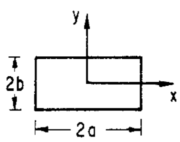
| geometry | $\langle v_z \rangle$ | $u$ | $D_h$ | $D_{eq}$ | $\Omega$ |
|---|---|---|---|---|---|
|  | $\left(-\frac{dP}{dz}\right)\frac{b^2}{3\mu}$ (A-1) | $\frac{3}{2}(1-\eta^2)$ (A-6) where $\eta = \frac{y}{b}$ (A-7) | $4b$ (A-11) | $4b\sqrt{\frac{2}{3}}$ (A-13) | $\frac{3}{2}$ (A-15) |
|  | $\left(-\frac{dP}{dz}\right)\frac{b^2}{3\mu}F$ (A-2) where $F = 1 - \frac{192}{R\pi^5}\sum_{j=1}^{\infty}\frac{\tanh(\lambda_j R)}{(2j-1)^5}$ (A-3) $\lambda_j = \frac{(2j-1)\pi}{2}$ (A-4) $R = \frac{a}{b}$ (A-5) | $\frac{3}{2F}\left[1-\eta^2-\frac{32}{\pi^3}\sum_{j=1}^{\infty}A_j\frac{\cosh(\lambda_j R\xi)}{\cosh(\lambda_j R)}\cos\lambda_j\eta\right]$ (A-8) where $A_j = \frac{(-1)^{j+1}}{(2j-1)^3}$ (A-9) $\xi = \frac{x}{a}$ (A-10) | $\frac{4b}{1+(1/R)}$ (A-12) | $4b\sqrt{\frac{2}{3}F}$ (A-14) | $\frac{3}{2[1+(1/R)]^2F}$ (A-16) |

Table II. Comparison of the Critical Reynolds Number Values with Those Calculated by Hanks and Ruo (1966)

| $R$ | $\bar{\xi}$ | $\bar{\eta}$ | $\bar{Z}$ | $\bar{F}$ | $Re_{h,c}$ this work | $Re_{h,c}$ Hanks and Ruo | $Re_{eq,c}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.637 | 3.434 | 0.422 | 1673 | 2060 | 1775 |
| 2.04 | 0 | 0.585 | 3.039 | 0.692 | 1547 | 1900 | 1566 |
| 2.36 | 0 | 0.582 | 2.904 | 0.733 | 1600 | 1960 | 1592 |
| 2.92 | 0 | 0.579 | 2.700 | 0.784 | 1706 | 2085 | 1656 |
| 3.92 | 0 | 0.578 | 2.438 | 0.839 | 1888 | 2315 | 1772 |
| ∞ | | 0.577 | 1.732 | 1.000 | 2800 | 2800 | 2285 |

parallel plates and rectangular ducts are given in Table I. Note that the rectangular duct geometry reduces to parallel plate as the aspect ratio, $R$, approaches infinity.

Hanks (1963) stated that, when $K_{max} = K_{max,c}$, then $Re = Re_c$, and he calculated $K_{max,c}$ as 404, a unique constant. Hence, the following equations are obtained from eq 11:

$$Re_{eq,c} = \frac{12928}{D_{eq}\bar{\phi}} \quad (13)$$

$$Re_{h,c} = \frac{12928\Omega}{D_h\bar{\phi}} \quad (14)$$

where $\bar{\phi}$ is the extremal value of $\phi$ evaluated at $x = \bar{x}$ and $y = \bar{y}$. According to Hanks (1963), "$\bar{x}$ and $\bar{y}$ are the coordinates of the spatial location within the cross section of the flow field which is least stable to a disturbance."

For flow between parallel plates, Hanks (1963) calculated $Re_{eq,c}$ as 2285. By use of eq 9, $Re_{h,c}$ can be easily obtained as 2800.

For flow in rectangular ducts, substitution of eq A-8, A-12, and A-16 in Table I into eq 14 yields

$$Re_{h,c} = \frac{4848R}{(1+R)F\bar{Z}} \quad (15)$$

where $\bar{Z}$ is the extremal value of $Z$ defined by

$$Z = u\left(\frac{1}{R^2}[\partial u/\partial\xi]^2 + [\partial u/\partial\eta]^2\right)^{1/2} \quad (16)$$

Note that eq 15 is exactly the same equation given by Hanks and Ruo (1966, eq 13). However, the function $F$, given by Hanks and Ruo,

$$F = 1 - \frac{192}{\pi^5}\sum_{j=1}^{\infty}\frac{\tanh(\lambda_j R)}{(2j-1)^{2j-1}} \quad (17)$$

is in error. The correct form is given by eq A-3 in Table I.

The values of $Re_{h,c}$ and $Re_{eq,c}$ are calculated by using eq 15 and 16 and are given in Table II. Although $\bar{\xi}$ and $\bar{\eta}$ values match the ones given by Hanks and Ruo (1966), their critical Reynolds number values are greater than those calculated in this work by a factor of $(3/2)^{1/2}$.

Table III. Values of $\bar{Z}$ for Selected Aspect Ratios

| $R$ | $m$ | $n$ | $\bar{Z}$ |
|---|---|---|---|
| 1 | 2.200 | 2.200 | 3.611 |
| 2.04 | 3.057 | 2.047 | 3.037 |
| 2.36 | 3.364 | 2.027 | 2.907 |
| 2.92 | 3.941 | 2.003 | 2.722 |
| 3.92 | 5.085 | 2.000 | 2.480 |

The calculation of $\bar{Z}$ using eq A-8 in Table I is rather tedious. For flow in rectangular ducts, Natarajan and Lakshmanan (1972) proposed the following approximate equation for the velocity distribution:

$$u = \left[\frac{m+1}{m}\right]\left[\frac{n+1}{n}\right](1-\xi^m)(1-\eta^n) \quad (18)$$

where

$$m = 1.7 + 0.5R^{1.4} \quad (19)$$

$$n = \begin{cases} 2 & \text{for } R \geq 3 \\ 2 + 0.3\left[\frac{1}{R} - \frac{1}{3}\right] & \text{for } R \leq 3 \end{cases} \quad (20)$$

The values of $\bar{Z}$, calculated by using eq 18, are given in Table III. Note that these values are very close to those calculated previously.

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# CORRESPONDENCE

## Letter to the Editor

*Sir:* In 1982, I published an article (Govind, 1982) in *Industrial & Engineering Chemistry Process Design and Development* that relied heavily on the writings of Fisher (1963), but the work by Fisher was not cited as a reference. The essential results in my work were original. The paper was subsequently the subject of a correspondence between Bitter (1983) and me (Govind, 1983), but again the Fisher article was not mentioned by either author.

An interested reader brought this matter to the attention of Hugh M. Hulburt, Editor of *Industrial & Engineering Chemistry Process Design and Development*, a short time before his death. Professor Hulburt wrote to me requesting a letter pointing out my error in not citing the earlier paper by Fisher (1963). This letter is presented to fulfill this decision, with which I agree completely.

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