CHAPTER 13
Approximate Solutions for 1-D Media

The Optically Thin Approximation

Gray medium between two diffuse gray isothermal parallel plates.
For $G(\tau)$ and $q(\tau)$ we obtained expressions (12.21) and (12.22)

$$G(\tau) = 2J_1E_2(\tau) + 2J_2E_2(\tau_L - \tau) + 2\pi\int_0^{\tau} S(\tau')E_1(\tau - \tau')\,d\tau' + 2\pi\int_{\tau}^{\tau_L} S(\tau')E_1(\tau' - \tau)\,d\tau', \qquad (13.1)$$

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau) + 2\pi\int_0^{\tau} S(\tau')E_2(\tau - \tau')\,d\tau' - 2\pi\int_{\tau}^{\tau_L} S(\tau')E_2(\tau' - \tau)\,d\tau', \qquad (13.2)$$

Assume optically thin medium: $\tau_L \ll 1$.
Evaluate $q(\tau)$ accurate up to $O(\tau)$ neglecting $O(\tau^2)$ terms.
Note: $E_3(x) = \frac{1}{2} - x + O(x^2)$ and $E_2(x) = 1 + O(x)$ from App.E. Note also that E_2 appears inside integrals.

$$q(\tau) \simeq J_1(1-2\tau) - J_2(1-2\tau_L+2\tau) + 2\pi\left[\int_0^{\tau} S(\tau')\,d\tau' - \int_{\tau}^{\tau_L} S(\tau')\,d\tau'\right]. \qquad (13.3)$$

Since $S(\tau)$ is in integral, we need to evaluate it $O(1)$. Note that for isotropic scattering $S(\tau)$ is given with Eq(12.5):

$$S(\tau) = (1-\omega)I_b(\tau) + \frac{\omega}{4\pi}G(\tau), \qquad (13.4)$$

Thus we need to evaluate $G(\tau)$ up to $O(1)$:

$$G(\tau) = 2J_1 + 2J_2 + O(\tau) \qquad (13.5)$$

Therefore $S(\tau') = (1-\omega)I_b(\tau') + \frac{\omega}{4\pi}(2J_1 + 2J_2) + O(\tau)$.

$$\Rightarrow \quad q(\tau) = J_1(1-2\tau) + J_2(1-2\tau) + (\omega J_1 + \omega J_2)[(\tau - 0) - (\tau_L - \tau)] + 2\pi(1-\omega)\left[\int_0^{\tau} I_b(\tau')d\tau' - \int_{\tau}^{\tau_L} I_b(\tau')d\tau'\right]$$

- If $I_b(\tau)$ is known:

$$q(\tau) = J_1[1 - 2\tau + \omega(2\tau - \tau_L)] - J_2[1 - 2\tau_L + 2\tau - 2\omega\tau + \omega\tau_L] + 2\pi(1-\omega)\left[\int_0^{\tau} I_b(\tau')d\tau' - \int_{\tau}^{\tau_L} I_b(\tau')d\tau'\right]$$

which leads to Eq(13.6):

$$q = J_1[1 - 2(1-\omega)\tau - \omega\tau_L] - J_2[1 + 2(1-\omega)\tau - (2-\omega)\tau_L] + 2\pi(1-\omega)\left[\int_0^{\tau} I_b(\tau')\,d\tau' - \int_{\tau}^{\tau_L} I_b(\tau')\,d\tau'\right]; \qquad (13.6)$$

- <u>If Radiative Equilibrium</u>:

  Gray medium $\Rightarrow \quad S = I_b = \dfrac{G}{4\pi} = \dfrac{J_1 - J_2}{2\pi}$, therefore Eq(13.6) becomes:

$$q = J_1[1-2(1-\omega)\tau - \omega\tau_L] - J_2[1+2(1-\omega)\tau-(2-\omega)\tau_L] + 2\pi(1-\omega)\frac{J_1+J_2}{2\pi}[(\tau-0)-(\tau_L-\tau)]$$

$$q = J_1[1 - \cancel{2(1-\omega)\tau} - \cancel{\omega}\tau_L + \cancel{2(1-\omega)\tau} - (1-\cancel{\omega})\tau_L]$$

$$- J_2[1 + \cancel{2(1-\omega)\tau} - (2-\cancel{\omega})\tau_L - \cancel{2(1-\omega)\tau} + (1-\cancel{\omega})\tau_L]$$

$$q = J_1(1-\tau_L) - J_2(1-\tau_L) = (J_1 - J_2)(1-\tau_L) = \text{const.}$$

If temperature of the medium specified, we are usually interested in divergence of heat flux (*dq/dz*). To obtain *dq/dz* accurate to $O(\tau)$, we must obtain $dq/d\tau$ to $O(1)$, thus $G(\tau)$ to $O(1)$. From Eq(12.24):

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G). \qquad (13.8)$$

Since $G(\tau) = 2J_1 + 2J_2 + O(\tau)$:

$$\frac{dq}{d\tau} = 2(1-\omega)(2\pi I_b - J_1 - J_2). \qquad (13.9)$$

<u>The Optically Thick Approximation (Diffusion Approx.)</u>

Optically thick slab: $\tau_L \gg 1$

In Eq(13.2) change integral variables $\tau' \to \tau'' = |\tau - \tau'|$

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau)$$
$$+ 2\pi\int_0^{\tau} S(\tau-\tau'')E_2(\tau'')\,d\tau'' - 2\pi\int_0^{\tau_L-\tau} S(\tau+\tau'')E_2(\tau'')\,d\tau''. \qquad (13.10)$$

We are large optical distance away from surfaces $\Rightarrow \quad \tau \gg 1$ and $\tau_L - \tau \gg 1$, therefore influence of J_1 and J_2 becomes negligible and we can replace integral limits with ∞ ($E_2(\tau'') = 0$ beyond actual limits).

$$q(\tau) \simeq 2\pi\int_0^{\infty} S(\tau-\tau'')E_2(\tau'')\,d\tau'' - 2\pi\int_0^{\infty} S(\tau+\tau'')E_2(\tau'')\,d\tau'', \qquad (13.11)$$

Taylor series expansion of *S*:

$$S(\tau \pm \tau'') = S(\tau) \pm \tau''\left(\frac{dS}{d\tau}\right)_\tau + \frac{(\tau'')^2}{2}\left(\frac{d^2S}{d\tau^2}\right)_\tau \pm \cdots.$$

Therefore:

$$\begin{aligned}
\frac{q(\tau)}{2\pi} &= S(\tau)\int_0^\infty E_2(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau'' E_2(\tau'')\,d\tau'' \\
&\qquad + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_2(\tau'')\,d\tau'' - + \cdots \\
&\quad - S(\tau)\int_0^\infty E_2(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau'' E_2(\tau'')\,d\tau'' \\
&\qquad - \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_2(\tau'')\,d\tau'' - - \cdots \\
&= -2\frac{dS}{d\tau}\int_0^\infty x E_2(x)\,dx + \mathcal{O}\left(\frac{1}{\tau^3}\right).
\end{aligned}$$

Note: Integration by parts:

$$\int_0^\infty x E_2(x)\,dx = -xE_3(x)\Big|_0^\infty + \int_0^\infty E_3(x)\,dx = -E_4(x)\Big|_0^\infty = \frac{1}{3},$$

and

$$q(\tau) = -\frac{4\pi}{3}\frac{dS}{d\tau}. \tag{13.12}$$

- For a non-scattering medium or a gray medium at radiative equilibrium: $S = I_b \quad \Rightarrow$

$$q(\tau) = -\frac{4\pi}{3}\frac{dI_b}{d\tau}. \tag{13.13}$$

- Isotropically scattering medium: We need to obtain $G(\tau)$ similar to $q(\tau)$:

$$\begin{aligned}
\frac{G(\tau)}{2\pi} &= S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau'' E_1(\tau'')\,d\tau'' \\
&\qquad + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_1(\tau'')\,d\tau'' - + \cdots \\
&\quad + S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' + \frac{dS}{d\tau}\int_0^\infty \tau'' E_1(\tau'')\,d\tau'' \\
&\qquad + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_1(\tau'')\,d\tau'' + + \cdots \\
&= 2S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' + \mathcal{O}\left(\frac{1}{\tau^2}\right) = 2S(\tau),
\end{aligned}$$

or

$$\frac{G}{4\pi} = S = (1-\omega)I_b + \omega\frac{G}{4\pi}$$

and

$$S(\tau) = \frac{G}{4\pi}(\tau) = I_b(\tau). \tag{13.14}$$

For an optically thick, isotropically scattering medium at radiative equilibrium or not:

$$q(\tau) = -\frac{4\pi}{3}\frac{dI_b}{d\tau}$$

Rosseland Approx. (Diffusion approx) On a spectral basis:

$$q_\eta = -\frac{4\pi}{3\beta_\eta}\frac{dI_{b\eta}}{dz}, \tag{13.15}$$

Total heat flux:

$$q = -\frac{4\sigma}{3\beta_R}\frac{d(n^2T^4)}{dz}, \tag{13.16}$$

where β_R is Rosseland mean extinction coefficient (defined in Eq(9.104)).

We may define a “radiative conductivity”

$$k_R = \frac{16n^2\sigma T^3}{3\beta_R}, \tag{13.17}$$

so that

$$q = -k_R\frac{dT}{dz}, \tag{13.18}$$

In 3-D:

$$\mathbf{q}_\eta = -\frac{4\pi}{3\beta_\eta}\nabla I_{b\eta}, \tag{13.19}$$

and

$$\mathbf{q} = -\frac{4\sigma}{3\beta_R}\nabla(n^2T^4) = -k_R\nabla T. \tag{13.20}$$

In practice, the method is useful only in optically extremely thick situations like heat transfer through hot glass.

- Deissler’s Jump BCs:

We took $\tau \gg 0$ and $\tau_L - \tau \gg 0$ for optically thick medium, so what happens close to surfaces?
No radiative principle states that the temperature of surface and adjacent mediu must be continuous.
For $\tau = 0$ Eq(13.10):

$$\begin{aligned} q(0) &= J_1 - 2\pi\int_0^\infty S(\tau'')E_2(\tau'')\,d\tau'' \\ &= J_1 - 2\pi\Big[S(0)\int_0^\infty E_2(\tau'')\,d\tau'' + \frac{dS}{d\tau}(0)\int_0^\infty \tau'' E_2(\tau'')\,d\tau'' \\ &\qquad + \frac{1}{2}\frac{d^2S}{d\tau^2}(0)\int_0^\infty (\tau'')^2E_2(\tau'')\,d\tau'' + \mathcal{O}\left(\frac{1}{\tau^3}\right)\Big]. \end{aligned}$$

Using equation (13.14) this expression becomes

$$q(0) = J_1 - \pi I_b(0) - \frac{2\pi}{3}\frac{dI_b}{d\tau}(0) - \frac{\pi}{2}\frac{d^2I_b}{d\tau^2}(0) + \mathcal{O}\left(\frac{1}{\tau^3}\right). \tag{13.23}$$

truncating the series after the second derivative.

$$q(\tau) = -\frac{4\pi}{3}\frac{dI_b}{d\tau}. \quad (13.13)$$

Substituting Eq(13.13)
(with the same order of accuracy)

$-\frac{4\pi}{3}\frac{dI_b}{d\tau}(0) = J_1 - \pi I_b(0) - \frac{2\pi}{3}\frac{dI_b}{d\tau}(0) - \frac{\pi}{2}\frac{d^2 I_b}{d\tau^2}(0)$, therefore

$$J_1 = \pi I_b(0) - \frac{2\pi}{3}\frac{dI_b}{d\tau}(0) + \frac{\pi}{2}\frac{d^2 I_b}{d\tau^2}(0). \quad (13.24)$$

For radiative equilibrium of a 1-D slab:

$$J_1 - \pi I_b(0) = -\frac{2\pi}{3}\frac{dI_b}{d\tau}(0) = \frac{1}{2}q(0) = \frac{1}{2}q, \quad (13.25)$$

since q=const. and therefore, $\frac{d^2 I_b}{d\tau^2} = 0$.

The generalized jump condition for multidimensional geometries:

$$J_w(\mathbf{r}_w) = \pi I_b(\mathbf{r}_w) - \frac{2\pi}{3}\frac{\partial I_b}{\partial \tau_z}(\mathbf{r}_w) + \frac{\pi}{4}\left(2\frac{\partial^2 I_b}{\partial \tau_z^2} + \frac{\partial^2 I_b}{\partial \tau_x^2} + \frac{\partial^2 I_b}{\partial \tau_y^2}\right)(\mathbf{r}_w), \quad (13.26)$$

<u>The Schuster-Schwarzschild Approximation (Two-flux approx):</u>

A very simple solution method for 1-D plane parallel slab.
Isotropically scattering gray medium: $\Phi = 1$.
From Eq(12.19):

$$\mu\frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}\int_{-1}^{+1} I\,d\mu, \quad -1 < \mu < +1. \quad (13.27)$$

Assume:

$$I(\tau,\mu) = \begin{cases} I^-(\tau), & -1 < \mu < 0, \\ I^+(\tau), & 0 < \mu < +1. \end{cases} \quad (13.28)$$

Substituting this in to Eq(13.27):

$$\mu\frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}(I^- + I^+). \quad (13.29)$$

Writing this as two space dependent equations (note I^- and I^+ are only $f(\tau)$) and integrating over upper and lower hemispheres:

$$\frac{1}{2}\frac{dI^+}{d\tau} = (1-\omega)I_b - I^+ + \frac{\omega}{2}(I^- + I^+), \quad (13.30a)$$

$$-\frac{1}{2}\frac{dI^-}{d\tau} = (1-\omega)I_b - I^- + \frac{\omega}{2}(I^- + I^+), \quad (13.30b)$$

with BCs:

$$\tau = 0: \quad I^+ = J_1/\pi, \quad (13.31a)$$

$$\tau = \tau_L: \quad I^- = J_2/\pi, \quad (13.31b)$$

From the definitions:

$$G = 2\pi \int_{-1}^{1} I \, d\mu = 2\pi(I^+ + I^-), \quad (13.32)$$

and

$$q = 2\pi \int_{-1}^{1} I \mu \, d\mu = \pi(I^+ - I^-). \quad (13.33)$$

Eq(13.30a)+ Eq(13.30b):

$$\frac{1}{2}\left(\frac{dI^+}{d\tau} - \frac{dI^-}{d\tau}\right) = 2(1-\omega)I_b - (I^+ + I^-) + \omega(I^- + I^+)$$

$$\frac{1}{2}\frac{1}{\pi}\frac{dq}{d\tau} = 2(1-\omega)I_b - (1-\omega)\frac{G}{2\pi}$$

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G)$$

Eq(13.30a) - Eq(13.30b):

$$\frac{1}{2}\left(\frac{dI^+}{d\tau} + \frac{dI^-}{d\tau}\right) = -(I^+ + I^-)$$

$$\frac{1}{2}\frac{1}{\pi}\frac{dG}{d\tau} = -\frac{1}{\pi}q$$

$$\frac{dG}{d\tau} = -4q$$

BCs:

$$\tau = 0: \quad I^+ = \frac{(I^+ + I^-) + (I^+ - I^-)}{2} = \frac{(G/2\pi) + (q/\pi)}{2} = \frac{G+2q}{4\pi} \quad \therefore \quad \frac{G+2q}{4\pi} = \frac{J_1}{\pi} \Rightarrow G + 2q = 4J_1$$

$$\tau = \tau_L: \quad I^- = \frac{(I^+ + I^-) - (I^+ - I^-)}{2} = \frac{G-2q}{4\pi} \quad \therefore \quad \frac{G-2q}{4\pi} = \frac{J_2}{\pi} \Rightarrow G - 2q = 4J_2$$

So we obtain ODEs together with necessary BCs for the problem.

See Example (13.4).

- Schuster-Schwarzschild always goes to correct optically thin limit ($\tau_L \to 0$)
- Method is easilky generalized by breaking up 4π to more than 2 components
  $\Rightarrow$ Discrete Ordinates Method (S_N approximation) see Ch15.

<u>The Milne-Eddington Approximation (Moment method) (Differential Approx.)</u>

Start with Eq(13.27):

$$\mu \frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}\int_{-1}^{+1} I \, d\mu, \quad -1 < \mu < +1. \quad (13.27)$$

Defining intensity moments: $I_k \equiv 2\pi \int_{-1}^{1} I \mu^k d\mu$.

For example: from zeroth moment integral in Eq(13.27): $\int_{-1}^{1} I d\mu = \frac{I_0}{2\pi}$.

Take the zeroth and first moment of Eq(13.27):
The zeroth:

$$2\pi\int_{-1}^{1}\mu\frac{dI}{d\tau}d\mu = 2\pi\int_{-1}^{1}(RHS)d\mu$$

$$\frac{dI_1}{d\tau} = (1-\omega)2\pi I_b\int_{-1}^{1}d\mu - 2\pi\int_{-1}^{1}Id\mu + \frac{\omega}{2}\frac{I_0}{2\pi}2\pi\int_{-1}^{1}d\mu$$

$$\frac{dI_1}{d\tau} = (1-\omega)4\pi I_b - I_0 + \omega I_0 = (1-\omega)(4\pi I_b - I_0)$$

The first:

$$2\pi\int_{-1}^{1}\mu^2\frac{dI}{d\tau}d\mu = 2\pi\int_{-1}^{1}\mu(RHS)d\mu$$

$$\frac{dI_2}{d\tau} = (1-\omega)2\pi I_b\int_{-1}^{1}\mu d\mu - 2\pi\int_{-1}^{1}I\mu d\mu + \frac{\omega}{2}\frac{I_0}{2\pi}2\pi\int_{-1}^{1}\mu d\mu$$

$$\frac{dI_2}{d\tau} = 0 - I_1 + 0$$

Note that: $\int_{-1}^{1}\mu d\mu = 0$.

Therefore:

$$\frac{dI_1}{d\tau} = (1-\omega)4\pi I_b - I_0 + \omega I_0 = (1-\omega)(4\pi I_b - I_0), \qquad (13.38)$$

$$\frac{dI_2}{d\tau} = -I_1, \qquad (13.39)$$

So we have 2 equations and 3 unknowns (I_0, I_1 and I_2). We need one more equation for closure (closing condition).
Assume the intensity to be isotropic over both the upper and lower hemispheres (like two-flux assumption):

$$I_k = 2\pi(I^-\underbrace{\int_{-1}^{0}\mu^k d\mu}_{\frac{(-1)^k}{k+1}} + I^+\underbrace{\int_{0}^{1}\mu^k d\mu}_{\frac{1}{k+1}}) = \frac{2\pi}{k+1}[(-1)^k I^- + I^+]$$

$$\therefore\ I_2 = \frac{2\pi}{3}[I^- + I^+] \text{ and } I_0 = 2\pi[I^- + I^+] \ \Rightarrow\ I_2 = \frac{I_0}{3}$$

Therefore 3 equations we obtained:

$$\frac{dI_1}{d\tau} = (1-\omega)(4\pi I_b - I_0)$$

$$\frac{dI_2}{d\tau} = -I_1$$

$$I_2 = \frac{I_0}{3}$$

With $G = I_0$ and $q = I_1$:

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G), \tag{13.42}$$

$$\frac{dG}{d\tau} = -3q. \tag{13.43}$$

BCs identical to two-flux BCs:

$$\tau = 0: \qquad G + 2q = 4J_1, \tag{13.44a}$$

$$\tau = \tau_L: \qquad G - 2q = 4J_2. \tag{13.44b}$$

- In case of radiative equilibrium: $\frac{dq}{d\tau} = 0$

  $\therefore \quad (1-\omega)(4\pi I_b - G) = 0 \quad \Rightarrow \quad G = 4\pi I_b$

  $$\frac{dG}{d\tau} = 4\pi \frac{dI_b}{d\tau} = -3q \quad \Rightarrow \quad q = -\frac{4\pi}{3}\frac{dI_b}{d\tau}$$

  which is same as diffusion approximation (optically thick).

Milne-Eddington may be generalized to higher order moments as well as more general geometries (moment method):

$$\begin{aligned} I(\mathbf{r}, \hat{\mathbf{s}}) &= I_0(\mathbf{r}) + I_{1x}(\mathbf{r})s_x + I_{1y}(\mathbf{r})s_y + I_{1z}(\mathbf{r})s_z + I_{2xx}(\mathbf{r})s_x^2 + I_{2xy}(\mathbf{r})s_x s_y + \cdots \\ &= I_0(\mathbf{r}) + \mathbf{I}_1(\mathbf{r}) \cdot \hat{\mathbf{s}} + \mathbf{I}_2(\mathbf{r}) : \hat{\mathbf{s}}\hat{\mathbf{s}} + \cdots. \end{aligned} \tag{13.46}$$

where $\mathbf{I_1}(\mathbf{r})$ is a vector (related to *q*)

$\mathbf{I_2}(\mathbf{r})$ is a second-rank tensor (which may be related to radiation pressure)

Direction cosines of unit direction vector $\hat{\mathbf{s}}$ are

$$s_x = \hat{\mathbf{s}} \cdot \hat{\mathbf{i}} = \sin\theta \cos\psi$$

$$s_y = \hat{\mathbf{s}} \cdot \hat{\mathbf{j}} = \sin\theta \sin\psi$$

$$s_z = \hat{\mathbf{s}} \cdot \hat{\mathbf{k}} = \cos\theta$$

Unknowns in Eq(13.46) are determined by taking the moments of the equation of transfer (i.e. by integrating over all directions after multiplication by $1, s_x, s_y, s_z, s_x^2, s_y^2, s_z^2, \ldots$). This is equivalent to the method of spherical harmonics (P_N Method of Ch14)